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Algebra embedding of $U_q(\mathrm{sl}(2))$ into the tensor product of two (q, h)-Weyl algebras

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ABSTRACT

In this paper we study relations between different (q, h)-Weyl algebras. We also show an embedding of the q-deformed universal enveloping algebra $U_q = U_q(\mathrm{sl}(2))$ into a tensor product of two (q, h)-Weyl algebras when $q \neq 1$. This allows us to study the limiting case $q \to 1$ in the so-called structure ladder of U_q.

1. Introduction

In [3] the (q, h)-deformation of the classical Weyl algebra was introduced. It turned out that this algebra incorporates features that allow us to treat continuous, h-discrete and q-discrete equations, orthogonal polynomials and special functions simultaneously on the algebraic level. We are convinced that most of basic properties of orthogonal polynomials and special functions are encoded in those algebras. How this idea is realized is shown in [3], where we studied the algebraic background for (q, h)-monomials and (q, h)-binomial functions, for example.

Recall that the classical Weyl algebra $W = W_{(1,0)}$ is the $\mathbb{C}$-algebra with a unit 1 generated by two elements D and X with a single relation

$$DX - XD = 1. \tag{1}$$

There is a standard representation on the space $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ (or polynomials, distributions, analytic, Schwartz functions) by differentiation (operator D) and multiplication by the variable x (operator X).

Abbreviations: SIE, scalar intrinsic endomorphisms.

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The deformed (q,h)-Weyl algebra $W_{(q,h)}$ is defined as the unital algebra over $\mathbb{C}$, generated by the operators X, S, S^{-1}, μ^{-1} subject to the relations

$$SX = (qX+h)S, \tag{2}$$
$$SS^{-1} = S^{-1}S = 1, \tag{3}$$
$$((q-1)X+h)\mu^{-1} = \mu^{-1}((q-1)X+h) = 1. \tag{4}$$

Within this article we assume that $q \in \mathbb{C}\setminus\{0\}$ and $h \in \mathbb{C}$ are fixed complex numbers. The limit $q \to 1$ is of special attention throughout the text. Also, depending on the representation that we consider, we might consider further restrictions, for instance, q and h are real numbers, q is positive and is not a unit root [3]. The first relation (2) is the essential one and it reflects the basic structure of $W_{(q,h)}$. The other two relations (3) and (4) only mean that the shift operator S and the so called graininess operator $\mu := (q-1)X+h$ are invertible.

There are many relations that can be derived from (2)–(4). We only mention the following relations:

$$\begin{aligned} SX - XS &= \mu S, & S^{-1}X - XS^{-1} &= -q^{-1}\mu S^{-1}, \\ S\mu &= q\mu S, & S^{-1}\mu &= q^{-1}\mu S^{-1}. \end{aligned} \tag{5}$$

It is convenient to introduce two more operators D (the central difference operator) and M (the mix operator) by using the following formulas (for $q \neq -1$):

$$D = \tfrac{q}{q+1}\mu^{-1}(S - S^{-1}), \tag{6}$$
$$M = \tfrac{q}{q+1}S + \tfrac{1}{q+1}S^{-1}. \tag{7}$$

Note that

$$S = M + \tfrac{\mu}{q}D,$$
$$S^{-1} = M - \mu D,$$
$$DX - XD = M, \tag{8}$$
$$MX - XM = \tfrac{q-1}{q}\mu M + \tfrac{1}{q}\mu^2 D. \tag{9}$$

There are several important representations of the (q,h)-deformed Weyl algebra [3]. We assume that a complex-valued function f is well-defined on a set $\{\sigma^j(x_0)\,|\,j \in \mathbb{Z}\} \subset \mathbb{C}$, where $\sigma(x) = qx+h$ and $x_0 \in \mathbb{C}\setminus\{\frac{-h}{q-1}\}$ is fixed [2]. The operators are given as follows:

$$\begin{aligned} Xf(x) &:= x\,f(x), \\ Sf(x) &:= f(\sigma(x)) = f(qx+h), \\ S^{-1}f(x) &:= f(\sigma^{-1}(x)) = f(\tfrac{x-h}{q}), \\ \mu(X)f(x) &:= ((q-1)x+h)\cdot f(x), \\ \mu(X)^{-1}f(x) &:= ((q-1)x+h)^{-1}\cdot f(x). \end{aligned} \tag{10}$$

In this paper we shall present explicitly the isomorphisms between different (q,h)-Weyl algebras. Next we shall show that the q-deformed universal enveloping algebra $U_q(\mathrm{sl}(2))$ is embedded into the tensor product of two (q,h)-Weyl algebras when $q \neq 1$. Then we shall review basic notions of ladders and analyze the so-called structure ladder of U_q. Since we will apply the general ladder theory [4], our approach will be more