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Extensions of set-theoretic solutions of the Yang–Baxter equation and a conjecture of Gateva-Ivanova ☆

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ABSTRACT

We develop a theory of extensions for involutive and nondegenerate solutions of the set-theoretic Yang–Baxter equation and use it to produce new families of solutions. As an application we construct an infinite family of counterexamples to a conjecture of Gateva-Ivanova related to the retractability of square-free solutions.

0. Introduction

The Yang–Baxter equation is one of the basic equations in mathematical-physics. For that reason, in [9] Drinfeld posed the question of finding set-theoretic solutions, i.e. bijective maps $r\colon X\times X\to X\times X$, where X is a set, satisfying

$$(r\times \mathrm{id})(\mathrm{id}\times r)(r\times \mathrm{id})=(\mathrm{id}\times r)(r\times \mathrm{id})(\mathrm{id}\times r).$$

Nondegenerate involutive solutions have received a lot of attention. Nondegenerate means that if one writes

$$r(x,y)=(\sigma_x(y),\tau_y(x)),\quad x,y\in X,$$

the maps σ_x and τ_x are bijective for each $x\in X$. Involutive means that $r^2=\mathrm{id}_{X\times X}$.

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The first results were obtained in the seminal works of Etingof, Schedler and Soloviev [10] and Gateva-Ivanova and Van den Bergh [17]. The theory of involutive solutions was further developed in several papers such as [11,12,18,19,13,3,15,14,4,5,8,7]. Most of these papers focus on the so-called square-free solutions, i.e. solutions satisfying $r(x,x)=(x,x)$ for all $x,y\in X$. Square-free solutions have several links with other topics (semigroups of I-type, semigroups of skew polynomial type, Bieberbach groups, quadratic algebras, etc.) and remarkable results were obtained. However, several important questions are still open. One of the main challenges is to construct new families of solutions. To understand how to construct new solutions, Etingof, Schedler and Soloviev considered an equivalence relation on X which induces a natural involutive solution $\operatorname{Ret}(X,r)$, the so-called retraction of (X,r). An involutive solution (X,r) is then called a multipermutation solution of level m if there exists a positive integer m such that $\operatorname{Ret}^m(X,r)$ has only one element, where the iterated retractions are defined inductively as $\operatorname{Ret}^k(X,r)=\operatorname{Ret}(\operatorname{Ret}^{k-1}(X,r))$ for $k>1$.

Multipermutation solutions has been intensively studied, see for example [4,5,13]. In this context, one of the most important unsolved problems is the following:

Conjecture *(Gateva-Ivanova). Every finite nondegenerate involutive square-free set-theoretic solution* $r: X\times X\to X\times X$ *such that* $|X|\geq 2$ *is a multipermutation solution.*

The conjecture was made in 2004, see [11, 2.28(I)].

The purpose of this note is to develop a theory of extensions of involutive nondegenerate set-theoretic solutions of the Yang–Baxter equation. To show the strength of the theory, we construct a set-theoretical solution that proves that [16, Conjecture 2.14] is not true and also answers [13, Open question 6.13(3)]. Another important application of our theory is the construction of counterexamples to the Gateva-Ivanova conjecture. One of these counterexamples is (up to isomorphism) the set-theoretic solution (X,r), where $X=\{1,2,\dots,8\}$ and

$$r(x,y)=(\varphi_x(y),\varphi_y(x)),\quad x,y\in X,$$

with

$$\begin{aligned}&\varphi_1=(57), &&\varphi_2=(68), &&\varphi_3=(26)(48)(57), &&\varphi_4=(15)(37)(68),\\ &\varphi_5=(13), &&\varphi_6=(24), &&\varphi_7=(13)(26)(48), &&\varphi_8=(15)(24)(37).\end{aligned}$$

Remarkably, extensions of this solution produce other counterexamples to the Gateva-Ivanova conjecture.

The paper is organized as follows. In Section 1 we review the basic notions of the set-theoretic Yang–Baxter equation. This section also includes the basics on cycle sets, which are nonassociative algebraic structures in bijective correspondence with nondegenerate involutive set-theoretic solutions. In Section 2 we develop the theory of extensions of cycle sets. Although these extensions are often hard to compute, they provide a very powerful tool to construct new solutions. Our main theorem appears in this section and states that if $X\to Y$ is a surjective homomorphism of cycle sets such that all the fibers have the same cardinality then X is an extension of Y. In Section 3 several examples of extensions are given. Among these examples one finds the semidirect product of cycle sets and several classes of extensions that are relatively easy to compute. We conclude the paper with counterexamples to the Gateva-Ivanova conjecture.

1. Preliminaries

Recall that a pair (X,r), where X is a nonempty set and

$$r\colon X\times X\to X\times X,\quad r(x,y)=(\sigma_x(y),\tau_y(x)),\quad x,y\in X,$$