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Nil-clean and strongly nil-clean rings

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ABSTRACT

An element a of a ring R is nil-clean if $a = e + b$ where $e^2 = e \in R$ and b is a nilpotent; if further $eb = be$, the element a is called strongly nil-clean. The ring R is called nil-clean (resp., strongly nil-clean) if each of its elements is nil-clean (resp., strongly nil-clean). It is proved that an element a is strongly nil-clean iff a is a sum of an idempotent and a unit that commute and $a - a^2$ is a nilpotent, and that a ring R is strongly nil-clean iff $R/J(R)$ is boolean and $J(R)$ is nil, where $J(R)$ denotes the Jacobson radical of R. The strong nil-cleanness of Morita contexts, formal matrix rings and group rings is discussed in details. A necessary and sufficient condition is obtained for an ideal I of R to have the property that R/I strongly nil-clean implies R is strongly nil-clean. Finally, responding to the question of when a matrix ring is nil-clean, we prove that the matrix ring over a 2-primal ring R is nil-clean iff $R/J(R)$ is boolean and $J(R)$ is nil, i.e., R is strongly nil-clean.

1. Introduction

There has been considerable interest in the structure of the rings whose elements are sums of certain special elements. For instance, a ring is called 2-good if every element is a sum of two units, while a ring is called (strongly) clean if every element is a sum of an idempotent and a unit (that commute with each other). Clean and strongly clean rings, and 2-good rings are active subjects, which can be traced back to Nicholson's work [25] in 1977 and Zelinsky's paper [31] in 1954, respectively. Not to mention, there are other important examples in the literature. This paper is concerned with two interesting variants of the clean property of rings, introduced by Diesl in [10]. Following Diesl [10], an element of a ring is called (strongly) nil-clean if it is a sum of an idempotent and a nilpotent (that commute with each other), and a ring is called (strongly) nil-clean if every element is (strongly) nil-clean. It comes as no surprise that nil-clean and strongly nil-clean rings are naturally connected to clean and strongly clean rings. Besides, the

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study of (strongly) nil-clean rings finds their close connections to strongly π-regular rings, boolean rings, and uniquely strongly clean rings. Furthermore, the nil-cleanness of a matrix ring is tightly linked to the famous Köthe Conjecture (see Section 6). The reader is referred to the papers [10,5,16,2] for the background and current stage of the study of nil-clean and strongly nil-clean rings.

We continue the study of nil-clean and strongly nil-clean rings with the focus on the structure and construction of strongly nil-clean rings and the question of when a matrix ring is nil-clean. We start by proving that strongly nil-clean elements are exactly those strongly clean elements a with $a - a^2$ nilpotent, with which some useful equivalent conditions of a strongly nil-clean element are obtained. These equivalent conditions are then used to prove the structure of a strongly nil-clean ring. This structure theorem, improving several results in [10], is utilized to conduct a detailed discussion regarding the strong nil-cleanness of Morita contexts, formal matrix rings and group rings in Sections 3 and 4, and the results obtained give various new families of strongly nil-clean rings. In Section 5, we go further to show the structure of a non-unital strongly nil-clean ring, and prove an extension theorem as an application. In Section 6, we prove that the matrix ring over a 2-primal ring R is nil-clean iff $R/J(R)$ is boolean and $J(R)$ is nil, i.e., R is a strongly nil-ring. This seems to be the best answer, so far, to the question of when a matrix ring is nil-clean.

Throughout, R is an associative ring with identity, and $C(R), J(R), U(R)$ and $\mathrm{Nil}(R)$ denote the center, the Jacobson radical, the unit group and the set of all nilpotent elements of R, respectively.

2. Strongly nil-clean rings

Let R be a ring and $a \in R$. If $a = e + b$ where $e^2 = e \in R$, $b \in R$ is a nilpotent and $eb = be$, then this expression is called a strongly nil-clean decomposition of a in R. Similarly, we define clean, nil-clean and strongly clean decompositions of an element. In this section, we give important equivalent conditions of a strongly nil-clean element, and prove the structure theorem of a strongly nil-clean ring. We also show that the so-called Jacobson Lemma holds for strongly nil-clean elements.

Theorem 2.1. *An element $a \in R$ is strongly nil-clean iff a is strongly clean in R and $a - a^2$ is a nilpotent.*

Proof. ($\Rightarrow$). Let $a = e + b$ be a strongly nil-clean decomposition in R. Then $a = (1 - e) + (2e - 1 + b)$ is a strongly clean decomposition in R. Moreover, $a^2 = e + 2eb + b^2$, so $a - a^2 = (1 - 2e - b)b$ is a nilpotent.

($\Leftarrow$). Let $a = e+u$ be a strongly clean decomposition in R and $a-a^2$ be a nilpotent. Then $a^2 = e+2eu+u^2$ and so $a - a^2 = (1 - 2e - u)u$. It follows that $1 - 2e - u$ is a nilpotent. So $a = (1 - e) + (-1 + 2e + u)$ is a strongly nil-clean decomposition in R. □

Corollary 2.2. *(See [10, Corollary 3.10].) A unit u of a ring R is strongly nil-clean iff $1 - u$ is a nilpotent.*

A uniquely strongly clean element of R is an element having a unique strongly clean decomposition in R. An element $a \in R$ is called strongly π-regular if $a^n \in Ra^{n+1} \cap a^{n+1}R$ for some positive integer n.

Lemma 2.3. *Let a be a strongly nil-clean element of R. Then:*

(1) *a has a unique strongly nil-clean decomposition in R.*
(2) *a is a strongly π-regular element of R.*
(3) *a is a uniquely strongly clean element of R.*

Proof. (1) is [10, Corollary 3.8]; (2) is by [10, p. 203, Remark].

(3) By Theorem 2.1, we know that a is strongly clean and a^2-a is a nilpotent. By the proof of Theorem 2.1, two different idempotents which give strongly clean decompositions of a must yield two different idempotents