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Generalized Clifford theory for graded spaces

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ABSTRACT

Assume that Γ is a finite abelian group and ε is an antisymmetric bicharacter on Γ. Let V be a Γ-graded space with a non-degenerate ε-symmetric bilinear form of degree zero. The goal of this paper is to develop a generalized Clifford theory on V. We first introduce the ε-Clifford algebra $C(V)$ and the ε-exterior algebra $\Lambda(V)$, and then establish an analogue of Chevalley identification between $C(V)$ and $\Lambda(V)$. Secondly, we extend the non-degenerate bilinear form of degree zero on V to a non-degenerate bilinear form on $\Lambda(V)$. Finally, as an application, we give a realization of the orthosymplectic ε-Lie algebra.

1. Introduction

Clifford and Weyl algebras are two of the most important algebraic objects in theoretical physics. These algebras represent creation and annihilation of particles satisfying Fermi or Bose statistics. Moreover, they also play an important role in many mathematical branches, such as differential geometry [1] and representation theory [2,3].

Clifford algebra naturally appears in many different ways. One natural approach is related to the algebraic theory of spinors [4, Chapter II]. Let V be a finite dimensional vector space with a symmetric bilinear form $(\cdot,\cdot)$. The Clifford algebra $C(V)$ is isomorphic to $T(V)/I$, where I is the ideal generated by the elements $x\otimes y+y\otimes x-2(x,y)\cdot 1$. Weyl algebra has a similar construction associated with a skew-symmetric bilinear form [5].

There are several generalized Clifford theories associated with "more general symmetry" [6–10]. In this paper, we develop a generalized Clifford theory on a Γ-graded space with a non-degenerate ε-symmetric

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bilinear form of degree zero, where Γ is a finite abelian group and ε is an antisymmetric bicharacter on Γ. For a field $\mathbb{F}$ of characteristic zero, let V be a Γ-graded $\mathbb{F}$-vector space with a non-degenerate ε-symmetric bilinear form of degree zero, and for any Γ-graded $\mathbb{F}$-vector space W, let $H(W)$ denote the set consisting of all homogeneous elements of W. The reader who is interested only in "super symmetry" may set $\Gamma = \mathbb{Z}_2$ and $\varepsilon(a,b) = (-1)^{ab}$.

This paper is organized as follows. In Section 2, we recall some basic facts about Γ-graded spaces and ε-symmetric bilinear forms. In Section 3, we define the ε-Clifford algebra $C(V)$ and the ε-exterior algebra $\Lambda(V)$ over V, and describe their graded structures. In Section 4, we establish an analogue of Chevalley identification η between the ε-Clifford algebra $C(V)$ and the ε-exterior algebra $\Lambda(V)$, and show that η satisfies the identity

$$\eta \circ \hat{\pi}_C = \hat{\pi}_\Lambda$$

on the subspace of skew ε-symmetric tensors $\Sigma(V)$ (see Theorem 4.8). In Section 5, we extend the non-degenerate bilinear form of degree zero on V to a non-degenerate bilinear form on $\Lambda(V)$, and show that $\iota_\Lambda(x)$ is the ε-adjoint operator of left ε-exterior multiplications $\epsilon_\Lambda(x)$ with respect to the extended bilinear form (see Theorem 5.4). In Section 6, we show that $\operatorname{ad} u$ is an ε-derivation of $\Lambda(V)$ for any $u \in H(\Lambda^2(V))$. Finally, we show that the subspace $\Lambda^2(V)$ is an ε-Lie algebra under the commutator (42), which is isomorphic to the orthosymplectic ε-Lie algebra $\mathfrak{osp}(V, \varepsilon)$ (see Theorem 6.3).

2. Preliminaries

An *antisymmetric bicharacter* ε on Γ is a map from $\Gamma \times \Gamma$ into the multiplicative group of nonzero elements in $\mathbb{F}$ satisfying

$$\varepsilon(a+b, c) = \varepsilon(a,c)\varepsilon(b,c), \tag{1}$$

$$\varepsilon(a, b+c) = \varepsilon(a,b)\varepsilon(a,c), \tag{2}$$

$$\varepsilon(a,b)\varepsilon(b,a) = 1 \tag{3}$$

for any $a, b, c \in \Gamma$. For an antisymmetric bicharacter ε on Γ, it is clear that $\varepsilon(a,a) = \pm 1$ for all $a \in \Gamma$. Define

$$\Gamma_{\bar{0}} = \{a \in \Gamma | \varepsilon(a,a) = 1\}, \quad \Gamma_{\bar{1}} = \{a \in \Gamma | \varepsilon(a,a) = -1\}.$$

We obtain

$$\varepsilon(a, 0) = \varepsilon(0, c) = 1 \tag{4}$$

for all $a, c \in \Gamma$ from identities (1) and (2) by setting $b = 0$. Hence

$$\varepsilon(a,b)\varepsilon(-a,b) = \varepsilon(0,b) = 1,$$

which implies that

$$\varepsilon(-a, b) = \varepsilon(b, a). \tag{5}$$

We also have

$$\varepsilon(-a, -b) = \varepsilon(b, -a) = \varepsilon(a, b). \tag{6}$$