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Connected quandles and transitive groups ☆

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ABSTRACT

We establish a canonical correspondence between connected quandles and certain configurations in transitive groups, called quandle envelopes. This correspondence allows us to efficiently enumerate connected quandles of small orders, and present new proofs concerning connected quandles of order p and $2p$. We also present a new characterization of connected quandles that are affine.

1. Introduction

1.1. Motivation

Let $Q = (Q, \cdot)$ be a set with a single binary operation. Then Q is a *rack* if all *right translations*

$$R_x : Q \to Q, \quad y \mapsto yx$$

are automorphisms of Q. If the rack Q is idempotent, that is, if $xx = x$ for all $x \in Q$, then Q is a *quandle*.

Consider the *right multiplication group*

$$\mathrm{RMlt}(Q) = \langle R_x : x \in Q \rangle,$$

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and note that Q is a rack if and only if $\mathrm{RMlt}(Q)$ is a subgroup of the automorphism group $\mathrm{Aut}(Q)$. A rack Q is said to be *connected* (also *algebraically connected* or *indecomposable*) if $\mathrm{RMlt}(Q)$ acts transitively on Q. The main subject of this work is connected quandles.

An important motivation for the study of quandles is the quest for computable invariants of knots and links. Connected quandles are of prime interest here because all colors used in a knot coloring fall into the same orbit of transitivity.

From a broader perspective, quandles are a special type of set-theoretical solutions to the quantum Yang–Baxter equation [10,12] and can be used to construct Hopf algebras [1]. There are indications, such as [13], that understanding racks and quandles, particularly the connected ones, is an important step towards understanding general set-theoretical solutions of the Yang–Baxter equation.

Our main result, Theorem 5.3, is a correspondence between connected quandles and certain configurations in transitive groups. Some variants of this representation were discovered independently in [11,15,24,36], but none of these works contains a complete characterization of the configurations as in Theorem 5.3, nor a discussion of the isomorphism problem as in Theorem 5.6. Using the correspondence, we reprove (and occasionally extend) several known results on connected quandles in a simpler and faster way. We focus on enumeration of “small” connected quandles, namely those of order less than 48 (see Section 8 and Algorithm 8.1) and those with p or $2p$ elements (see Section 9). Our proof of non-existence of connected quandles with $2p$ elements, for any prime $p > 5$, is based on a new group-theoretical result for transitive groups of degree $2p$, Theorem 10.1.

The modern theory of quandles originated with Joyce’s paper [24] and the introduction of the knot quandle, a complete invariant of oriented knots. Subsequently, quandles have been used as the basis of various knot invariants [4–6] and in algorithms on knot recognition [6,14].

But the roots of quandle theory are much older, going back to self-distributive quasigroups, or *Latin quandles* in today’s terminology, see [38] for a comprehensive survey of results on Latin quandles and their relation to the modern theory. Another vein of results has been motivated by the abstract properties of reflections on differentiable manifolds [27,30], resulting in what is now called *involutory quandles* [39]. Yet another source of historical examples is furnished by conjugation in groups, which eventually led to the discovery of the above-mentioned knot quandle by Joyce and Matveev [24,31].

Quandles have also been studied as algebraic objects in their own right, and we will now briefly summarize the most relevant results. Every quandle decomposes into orbits of transitivity of the natural action of its right multiplication group. An attempt to understand the orbit decomposition was made in [11,34], and a full description has been obtained in two special cases: for medial quandles [23] and for involutory quandles [36]. The orbits are not necessarily connected, but they share certain properties with connected quandles.

There have been several attempts to understand the structure of connected quandles, see e.g. [1]. In our opinion, the homogeneous representation reviewed in Section 3 is most useful in this regard. It was introduced by Galkin and Joyce [15,24], and led to several structural and enumeration results, such as [13,17,41]. Some of them will be presented in Sections 8 and 9. A classification of simple quandles can be found in [1,25].

1.2. Summary of results

The paper is written as a self-contained introduction to connected quandles. Therefore, in the next two sections, we review the theory necessary for proving the main result. Although the opening sections contain no original ideas, our presentation is substantially different from other sources. We prove the main result in Section 5, and the rest of the paper is concerned with its applications.

In Section 2 we develop basic properties of quandles in relation to the right multiplication group and its derived subgroup. In Section 3 we introduce the homogeneous representation (Construction 3.1) and characterize homogeneous quandles as precisely those obtained by this construction (Theorem 3.6). In