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Reduced torsion pairs

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ABSTRACT

Let H be a finite dimensional hereditary algebra of infinite representation type over an algebraically closed field. A torsion pair $(\mathcal{T}, \mathcal{F})$ in H-mod is called *reduced* if there are no indecomposable Ext-projective modules in $\mathcal{T}$ as well as no nonzero Ext-injective modules in $\mathcal{F}$. An anti-chain of reduced torsion pairs of width card K will be constructed.

0. Introduction

Let $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ be a finite connected quiver without oriented cycles. The set of vertices $\mathcal{Q}_0$ is labelled by $\{1, \ldots, n\}$. For an algebraically closed field K we consider the category $\text{rep}_K \mathcal{Q}$ of finite dimensional K-linear representations of $\mathcal{Q}$. This is an abelian length category, and equivalent to the module category H-mod of finite dimensional modules over the path algebra $H = K\mathcal{Q}$. In the study of module categories or more generally e.g. of triangulated categories with t-structure, see [5, Chap. II], frequently and successfully torsion pairs are used:

If $\mathcal{T}$ and $\mathcal{F}$ are classes of modules in H-mod, the pair $(\mathcal{T}, \mathcal{F})$ is called a torsion pair, provided

(a) the torsion class $\mathcal{T}$ as well as the torsion free class $\mathcal{F}$ is closed under extensions. $\mathcal{T}$ is closed under factors, $\mathcal{F}$ is closed under subobjects.

(b) For $T \in \mathcal{T}$ and $F \in \mathcal{F}$ we have $\text{Hom}_H(T, F) = 0$.

(c) For a module $M \in H$-mod there exists a short exact sequence

$$0 \to tM \to M \to fM \to 0,$$

with $tM \in \mathcal{T}$ and $fM = M/tM \in \mathcal{F}$. This short exact sequence is unique, and it is called the *canonical short exact sequence* of M with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. The module tM is called the *torsion submodule* of M.

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If $\mathcal{T}$ is a torsion class, i.e. closed under extensions and factors, then $(\mathcal{T}, \mathcal{F})$ with $\mathcal{F} = \{Y \,|\, \mathrm{Hom}(\mathcal{T}, Y) = 0\}$ is a torsion pair. Dually, if $\mathcal{F}$ is a torsion free class, there exists a unique torsion class $\mathcal{T}$ making $(\mathcal{T}, \mathcal{F})$ a torsion pair.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in H-mod, a module $X \in \mathcal{T}$ ($Y \in \mathcal{F}$) is called Ext-*projective* in $\mathcal{T}$ (Ext-*injective in* $\mathcal{F}$), if $\mathrm{Ext}(X, \mathcal{T}) = 0$ ($\mathrm{Ext}(\mathcal{F}, Y) = 0$, respectively). Clearly a projective module $P \in \mathcal{T}$ is Ext-projective (in $\mathcal{T}$). If $X \in \mathcal{T}$ is indecomposable and not projective then X is Ext-projective in $\mathcal{T}$ if and only if $\tau_H X \in \mathcal{F}$ and is Ext-injective. Here τ_H denotes the Auslander–Reiten translation in H-mod.

If X is Ext-projective in $\mathcal{T}$ with r pairwise nonisomorphic indecomposable direct summand, then X is a partial tilting module in H-mod. The class $\mathrm{Gen}(X)$ of modules generated by a partial tilting H-module X is the smallest torsion class in H-mod containing X as an Ext-projective module. The *right perpendicular category* $X^{\perp}$ of a partial tilting module X, defined by the modules M with $\mathrm{Hom}(X, M) = 0$ and $\mathrm{Ext}(X, M) = 0$ is equivalent to C-mod, where C is a (not necessarily connected) hereditary algebra with $n - r$ simple modules. To simplify notations we will identify these two equivalent categories.

It was shown in [1] that the attachment $(\mathcal{T}, \mathcal{F}) \mapsto (\mathcal{T} \cap X^{\perp}, \mathcal{F} \cap X^{\perp})$ defines a bijection between the torsion pairs $(\mathcal{T}, \mathcal{F})$ in H-mod, containing $X \in \mathcal{T}$ as an Ext-projective module, and all torsion pairs in C-mod. Clearly the torsion class $\mathcal{T} \cap X^{\perp}$ may contain nonzero Ext-projective modules, even if $X \in \mathcal{T}$ is chosen maximal. For a torsion pair $(\mathcal{T}', \mathcal{F}')$ in C-mod consider the class $\mathcal{T}$ of H-modules M, which are the middle term of short exact sequences of the form $0 \to G \to M \to Y' \to 0$, where G is generated by X and $Y' \in \mathcal{T}'$. Then $\mathcal{T}$ is a torsion class in H-mod, $X \in \mathcal{T}$ is Ext-projective in $\mathcal{T}$ and $\mathcal{T} \cap X^{\perp} = \mathcal{T}'$. If $\mathcal{T}' = \mathcal{T}_0 \cap X^{\perp}$ for some torsion class $\mathcal{T}_0$ in H-mod containing X as Ext-projective object, by this construction one gets back the original torsion class $\mathcal{T}_0$. If for example the algebra C is representation finite, then there are only finitely many torsion pairs in C-mod, hence there exist only finitely many torsion pairs in H-mod containing X as Ext-projective torsion module. For details see [1]. The analogue reduction procedure works for Ext-injective modules $Y \in \mathcal{F}$. Iterating these reductions, it stops after finitely many steps and one gets a “final” torsion pair $(\hat{\mathcal{T}}, \hat{\mathcal{F}})$ in $\hat{C}$-mod:

Either $\hat{C} = 0$, or $\hat{C} \neq 0$ but still hereditary, $\hat{\mathcal{T}}$ contains no indecomposable Ext-projective module and $\hat{\mathcal{F}}$ contains no nonzero Ext-injective module. Such a torsion pair is called a *reduced* torsion pair.

The set of torsion pairs in H-mod form a poset (even a lattice), if one defines $(\mathcal{T}_1, \mathcal{F}_1) \leq (\mathcal{T}_2, \mathcal{F}_2)$ if $\mathcal{T}_1 \subset \mathcal{T}_2$, see e.g. [1]. An anti-chain in a poset consists of incomparable elements.

The main result of the paper is

Theorem. *Let H be a finite dimensional connected representation infinite hereditary algebra over an algebraically closed field K. Then there exists an anti-chain of reduced torsion pairs in H-*mod *of width* $\mathrm{card}\, K$.

In the last section we will relate reduced torsion pairs in H-mod to tilting and cotilting modules in H-Mod, the category of all H-modules.

For unexplained terminology and results we refer to the monographs [2,11,13], or to the survey [6].

1. Preliminaries

Since H is hereditary, the Auslander–Reiten translation $\tau = \tau_H$ is the left exact functor $\tau = D\,\mathrm{Ext}(-, H)$ where $D = \mathrm{Hom}_K(-, K)$ is the usual duality. The functor $\tau^- = \mathrm{Ext}(DH, -)$ is right exact. Since H is connected, the Auslander–Reiten quiver $\Gamma(H)$ of H-mod consists in the case when H is representation finite of exactly one component. In this case $\Gamma(H)$ is the preprojective and preinjective component, simultaneously.

If H is representation infinite and connected, the Auslander–Reiten quiver $\Gamma(H)$ contains exactly one infinite preprojective component $\mathcal{P}$, containing all indecomposable projective modules, exactly one infinite preinjective component $\mathcal{I}$, containing all indecomposable injective modules, and infinitely many regular components. If H is tame, all regular components are tubes; in the wild case, the regular components are