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Star operations on overrings of Noetherian domains

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ABSTRACT

Let R be a Noetherian domain, and let $\mathrm{Star}(R)$ denote the set of star operations on R. Call R *star regular* if $|\mathrm{Star}(T)| \le |\mathrm{Star}(R)|$ for each overring T of R. In the case where $\mathrm{Star}(R)$ is finite we show that star regularity becomes a local property, and, further assuming that R is one-dimensional and local with infinite residue field, we prove that R is star regular. We also study the question of whether finiteness of $\mathrm{Star}(T)$ for each proper overring of a one-dimensional Noetherian domain R implies finiteness of $\mathrm{Star}(R)$.

0. Introduction

Let R be an integral domain with quotient field K, and let $\mathcal{F}(R)$ be the set of nonzero fractional ideals of R. A mapping $* : \mathcal{F}(R) \to \mathcal{F}(R)$, $E \mapsto E^*$, is called a *star operation* on R if the following conditions hold for all $a \in K \setminus \{0\}$ and $E, F \in \mathcal{F}(R)$:

(I) $(aE)^* = aE^*$ and $R^* = R$;
(II) $E \subseteq E^*$; if $E \subseteq F$, then $E^* \subseteq F^*$; and
(III) $(E^*)^* = E^*$.

The simplest star operations are the d-operation, defined by $E^d = E$ for every $E \in \mathcal{F}(R)$, and the v-operation defined by $E^v = (R : (R : E))$ for every $E \in \mathcal{F}(R)$. We denote by $\mathrm{Star}(R)$ the set of all star operations on R, and we say that R is a *divisorial* domain if $d = v$, equivalently, $\mathrm{Star}(R) = \{d\}$, equivalently, $|\mathrm{Star}(R)| = 1$.

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For several years, motivated by well-known characterizations of integrally closed and Noetherian divisorial domains [3,8], we have been interested in domains admitting only finitely many star operations [4–7]. Our investigations have often involved examination of the star operations on overrings of the base ring (that is, rings between the base ring and its quotient field), and the goal of this paper is to study, for a Noetherian domain R, how $|\mathrm{Star}(R)|$ affects $|\mathrm{Star}(T)|$ for each overring T of R, and conversely, with emphasis on the case where $\mathrm{Star}(R)$ is finite.

Call a domain R *star regular* if $|\mathrm{Star}(R)| \geq |\mathrm{Star}(T)|$ for each overring T of R. Since for (non-field) Noetherian domains R, finiteness of $\mathrm{Star}(R)$ implies that the (Krull) dimension of R is one [5, Theorem 2.1], we usually impose this restriction. In Section 1 we give a simple example of a one-dimensional local Noetherian domain R that fails miserably to be star regular; indeed, for this R, we have $|\mathrm{Star}(R)| = 1$ but $|\mathrm{Star}(T)| = \infty$ for some overring T. We show, for a one-dimensional Noetherian domain R, that if R is locally star regular, then it is star regular, and we prove the converse in the case $|\mathrm{Star}(R)| < \infty$ (and give an example showing that the converse does not hold without the finiteness assumption). We conjecture that if R is a local Noetherian domain with $1 < |\mathrm{Star}(R)| < \infty$, then R is star regular. The main result of the paper is that the conjecture holds when R has infinite residue field (Corollary 1.18). The proof relies heavily on the characterization in [7] of local Noetherian domains that have infinite residue field and only finitely many star operations.

In Section 2, we consider the question of whether finiteness of $\mathrm{Star}(T)$ for each proper overring of a Noetherian domain R implies finiteness of $\mathrm{Star}(R)$. We show in Theorem 2.2 that this does occur when R is *non*-local, and in Theorems 2.3 and 2.4, we show that if (R, M) is a local Noetherian domain for which each proper overring of R admits only finitely many star operations, then $\dim R = 1$, and, further assuming that R/M is infinite, that $(R : M)$ is either a PID with at most four maximal ideals or $(R : M)$ has two maximal ideals, both of whose residue fields are isomorphic to R/M. This is as close to a characterization of the situation as we come, and we close with several classes of examples of local Noetherian domains (R, M) (with infinite residue field) for which $|\mathrm{Star}(R)| = \infty$ but $|\mathrm{Star}(T)| < \infty$ for each proper overring T of R.

1. Star regularity

We recall the definition:

Definition 1.1. A domain R is *star regular* if $|\mathrm{Star}(R)| \geq |\mathrm{Star}(T)|$ for each overring T of R.

Since we shall refer to it several times, we restate [5, Theorem 2.3].

Lemma 1.2. *For a Noetherian domain R,* $|\mathrm{Star}(R)| = \prod_{M \in \mathrm{Max}(R)} |\mathrm{Star}(R_M)|$.

We remark that the proof of this result in [5] did not distinguish cardinalities in the infinite case. However, [2, Theorem 2.3] is a generalization of Lemma 1.2, and there the proof is valid for all cardinalities.

We begin with an example of a local Noetherian divisorial domain R that not only fails to be star regular but that has an overring T satisfying $|\mathrm{Star}(T)| = \infty$.

Example 1.3. Let k be a field, $D = k[X^3, X^7]$, and $P = (X^3, X^7)D$. Then $P^{-1} = k[X^3, X^7, X^{11}] = D + DX^{11}$. Now let $R = D_P$ and $M = PD_P$. Then $M^{-1} = P^{-1}D_P$, and $\dim_k M^{-1}/M = 2$. Thus $|\mathrm{Star}(R)| = 1$. If $N = (X^3, X^7, X^{11})M^{-1}$, then $S_0 := (M^{-1} : N) = (N : N) = k[X^3, X^4]_{(X^3,X^4)k[X^3,X^4]} = M^{-1} + M^{-1}X^4 + M^{-1}X^8$. It follows that S_0 is singly generated as an M^{-1}-algebra, and it is easy to see that $\dim_k S_0/N = 3$. If Q denotes the maximal ideal of S_0, then $S_1 := (S_0 : Q) = (Q : Q) = k[X^3, X^4, X^5]_{(X^3,X^4,X^5)k[X^3,X^4,X^5]}$. Then, since S_1 is not a PID and $\dim_k S_1/NS_1 = 3$, [7, Proposition 2.8] yields $|\mathrm{Star}(M^{-1})| \geq |k| + 2$. In particular, $|\mathrm{Star}(M^{-1})| = \infty$ if k is infinite. (It is easy to see that we also have $|\mathrm{Star}(D)| = 1$ and $|\mathrm{Star}(P^{-1})| = \infty$ if $|k| = \infty$. Since we do not need this, we omit the details.)