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On tame algebras of semiregular tubular type ☆

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ABSTRACT

We describe the structure of tame finite dimensional algebras over an algebraically closed field having strongly simply connected Galois coverings and the Auslander–Reiten quivers consisting of semiregular tubes.

1. Introduction and the main results

Throughout the paper by an algebra we mean a basic, indecomposable, finite dimensional K-algebra over an algebraically closed field K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite dimensional (over K) right A-modules and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules. It follows from general theory that every algebra A is isomorphic to a bound quiver algebra KQ/I, where $Q = Q_A$ is the Gabriel quiver of A and I is an admissible ideal in the path algebra KQ of Q over K (see [1, Chapter II]). Moreover, for a bound quiver algebra $A = KQ/I$, the category $\operatorname{mod} A$ is equivalent to the category $\operatorname{rep}_K(Q, I)$ of finite dimensional representations of Q over K bounded by I (see [1, Chapter III]).

From the remarkable Tame and Wild Theorem of Drozd [20] (see also [15]) the class of finite dimensional algebras over an algebraically closed field K may be divided into two disjoint classes. The first class is formed by the tame algebras for which the indecomposable modules occur in each dimension in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite dimensional algebras

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over K. Accordingly, we may realistically hope to classify the indecomposable finite dimensional modules only for the tame algebras. More precisely, a finite dimensional K-algebra over an algebraically closed field K is called *tame*, if for any dimension d, there exists a finite number of $K[x]$–A-bimodules M_i, $1 \le i \le n_d$, which are free of finite rank as left modules over the polynomial algebra $K[x]$ in one variable and all but finitely many isomorphism classes of modules in $\operatorname{ind} A$ of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some i. Moreover, let $\mu_A(d)$ be the least number of $K[x]$–A-bimodules satisfying the above condition for d. Then A is said to be of *polynomial growth* (respectively, *domestic*) if there exists a positive integer m such that $\mu_A(d) \le d^m$ (respectively, $\mu_A(d) \le m$) for any $d \ge 1$ (see [16,40]).

An important combinatorial and homological invariant of an algebra A is its Auslander–Reiten quiver Γ_A whose vertices are the isomorphism classes of modules in $\operatorname{ind} A$, the arrows correspond to the irreducible homomorphisms between modules in $\operatorname{ind} A$, and we have the Auslander–Reiten translations $\tau_A = D\operatorname{Tr}$ and $\tau_A^{-1} = \operatorname{Tr} D$ related to almost split sequences in $\operatorname{mod} A$ (see [1, Chapter IV]). We do not distinguish between a module in $\operatorname{ind} A$ and the vertex of Γ_A corresponding to it. By a component of Γ_A we mean a connected component of the translation quiver Γ_A. A component $\mathcal{C}$ of Γ_A is called *regular* if $\mathcal{C}$ contains neither a projective module nor an injective module, and *semiregular* if $\mathcal{C}$ does not contain both a projective module and an injective module. The shapes of regular and semiregular components of the Auslander–Reiten quiver Γ_A of an algebra A have been described by Liu in [27,28] and Zhang [51] (regular components). In particular, it is known that a regular component $\mathcal{C}$ of Γ_A contains an oriented cycle if and only if $\mathcal{C}$ is a stable tube, that is, a component of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \ge 1$ (see [27,51]). Moreover, it has been proved in [28] that a semiregular component $\mathcal{C}$ of Γ_A contains an oriented cycle if and only if $\mathcal{C}$ is a semiregular tube, that is, a ray tube (obtained from a stable tube by a finite number of ray insertions) or a coray tube (obtained from a stable tube by a finite number of coray insertions). An algebra A is said to be of *semiregular type* (respectively, of *semiregular tubular type*) if all components of Γ_A are semiregular (respectively, are semiregular tubes). The class of algebras of semiregular type consists of algebras of infinite representation type and contains the following distinguished classes of algebras: the hereditary algebras of infinite representation type (see [1]), the tilted algebras with semiregular connecting components (see [1,23, 35–38]), the tubular algebras [35], the canonical algebras [35], and the quasitilted algebras of canonical type (see [14,26,47]). It would be interesting to find a description of all algebras of semiregular type. We refer to the recent papers [6] and [22] describing interesting classes of tame algebras of semiregular type.

Frequently, applying covering techniques, we may reduce the representation theory of a given tame algebra to that for the corresponding tame simply connected algebras. Recall that, following [2], an algebra A is called *simply connected* if A is triangular (the Gabriel quiver Q_A of A is acyclic) and, for any presentation $A \cong KQ/I$ of A as a bound quiver algebra, the fundamental group $\Pi_1(Q, I)$ of (Q, I) is trivial. Further, following [41], an algebra A is called *strongly simply connected* if every convex subcategory of A is simply connected. It has been proved in [41, Theorem 4.1] that an algebra A is strongly simply connected if and only if the first Hochschild cohomology group $H^1(C, C)$ vanishes for every convex subcategory C of A, and if and only if every convex subcategory C of A satisfies the separation condition of Bautista, Larrión and Salmerón [3]. We also note that, for a representation-finite algebra A, A is simply connected if and only if A is strongly simply connected, and if and only if the fundamental group $\Pi_1(|\Gamma_A|)$ of the topological realization $|\Gamma_A|$ of the Auslander–Reiten quiver Γ_A of A vanishes (see [11,12]). Finally, an indecomposable triangular locally bounded category $R = KQ/I$ is said to be *strongly simply connected* if the following two conditions are satisfied: (1) for any vertices x and y in Q there are only finitely many paths in Q from x to y (R is interval-finite in the sense of [12]); (2) every finite convex subcategory C of R is strongly simply connected.

We also mention that, by a recent result of Brüstle, de la Peña and Skowroński [13], a strongly simply connected algebra is tame if and only if its Tits quadratic form is weakly nonnegative. Moreover, by a result of Kasjan [25], the tame strongly simply connected algebras form an open scheme. We refer also to [31,33,45]