Journal of Pure and Applied Algebra 219 (2015) 1564–1591

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

The characterization by automata of certain profinite groups

Adam Woryna

Silesian University of Technology, Institute of Mathematics, ul. Kaszubska 23, 44-100 Gliwice, Poland

ARTICLE INFO

Article history:
Received 18 March 2014
Available online 14 June 2014
Communicated by G. Rosolini

MSC:
20B27; 20B40; 20F29; 20E22; 20E18; 20E08

ABSTRACT

We use the combinatorial language of automata to define and study profinite groups which are infinitely iterated permutational wreath products of transitive finite permutation groups. We provide some naturally defined automaton realizations of these groups by the so-called time-varying automata, as well as by Mealy automata to characterize these groups as automaton groups, i.e. as groups topologically generated by a single automaton.

1. Introduction

Let (G, X) and (H, Y) be permutation groups on the sets X and Y, respectively. Let H^X be the direct power of copies of H indexed by X with the elements written as functions $f: X \to H$. For every $f \in H^X$ and every $g \in G$ the pair (f, g) defines a permutation of the cartesian product $X \times Y$ of the sets X and Y in the following way:

$$(f,g)\big((x,y)\big) = \big(g(x), h(y)\big), \quad (x,y) \in X \times Y,$$

where $h = f(x) \in H$ (through the whole paper we reserve the notation $f(x)$ for the result of applying a function f to an object x). The set of all such permutations (i.e. permutations of $X \times Y$ corresponding to all pairs $(f, g) \in H^X \times G$) forms a permutation group on $X \times Y$ which is called the permutational wreath product of the groups (G, X) and (H, Y) and is denoted by $H \wr_X G$. The use of the G-set X in the above notation is justified, as the structure of the permutational wreath product $H \wr_X G$ depends on this set. For example, if G, X and H are finite, then the group $H \wr_X G$ is also finite, and its order is equal to $|H|^{|X|} \cdot |G|$. Note that if π and π' are permutations of the set $X \times Y$ defined by the pairs (f, g) and (f', g'), respectively, then the composition $\pi \circ \pi'$ is defined by the pair (f'', g''), where $g'' = g \circ g'$, $f'' = f \circ f'_g$ and $f'_g = g \circ f' \in H^X$ (in the paper we use the right action convention for composition of mappings; in particular, we have $g \circ g'(x) = g'(g(x))$ for any $g, g' \in G$ and $x \in X$). This reveals the permutational wreath

E-mail address: adam.woryna@polsl.pl.

http://dx.doi.org/10.1016/j.jpaa.2014.06.016
0022-4049/

product $H \wr_X G$ as a group isomorphic with the semidirect product $H^X \rtimes G$, where G acts on the direct power H^X by permuting the factors. This construction is associative, i.e. if (K, Z) is a permutation group on the set Z, then both the permutational wreath product $(K \wr_Y H) \wr_X G$ and the permutational wreath product $K \wr_{X\times Y} (H \wr_X G)$ are precisely the same permutation group on the set $X \times Y \times Z$. For more about permutational wreath products see [22] (see also Section 5).

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers and let $(G_i, X_i)_{i\in\mathbb{N}}$ be an arbitrary sequence of permutation groups. For every $i \in \mathbb{N}$ we define the iterated permutational wreath product $W_i = \wr_{k=1}^{i} G_k$ of the first i groups as a permutation group on the cartesian product $X^{(i)} = X_1 \times \ldots \times X_i$ as follows:

$$W_1 := G_1, \qquad W_{i+1} := G_{i+1} \wr_{X^{(i)}} W_i, \quad i \geq 1.$$

For every $i \in \mathbb{N}$ the mapping $(f, g) \mapsto g$, where $f \in (G_{i+1})^{X^{(i)}}$, $g \in W_i$, defines a homomorphism $\phi_i: W_{i+1} \to W_i$, and the sequence $(W_i, \phi_i)_{i\in\mathbb{N}}$ forms an inverse system. In the present paper we consider the inverse limit

$$W_\infty = \wr_{i=1}^{\infty} G_i := \varprojlim_i W_i = \varprojlim_i \wr_{k=1}^{i} G_k \tag{1}$$

of such an inverse system and call the *infinitely iterated permutational wreath product of the sequence* $(G_i, X_i)_{i\in\mathbb{N}}$. Thus, according to the definition of the inverse limit, the group W_∞ consists of all sequences $(h_i)_{i\in\mathbb{N}}$ from the infinite cartesian product $\prod_{i\in\mathbb{N}} W_i$ of the groups $W_i = \wr_{k=1}^{i} G_k$ which satisfy the following condition: $\phi_i(h_{i+1}) = h_i$ for each $i \in \mathbb{N}$. If the sets X_i are all finite, then each W_i is a finite group and we obtain W_∞ as an inverse limit of finite groups, that is as a profinite group.

If each set X_i is finite and contains at least two elements, then it is usually called a *finite alphabet* and its elements are called *letters*. The arising sequence $X = (X_i)_{i\in\mathbb{N}}$ (further called *a changing alphabet*) defines a rooted tree X^*, which is an example of a spherically homogeneous rooted tree (see Section 4.1). The tree X^* consists of finite words which are sequences from finite cartesian products of the form $X^{(t)} := X_1 \times \ldots \times X_t$ ($t \in \mathbb{N}$), and the automorphism group $Aut(X^*)$ is an example of a topological group with the topology induced by a metric δ in which two automorphisms are *close* if they act in the same way on the set $X^{(t)}$ for a *large* number t. Specifically, we can define δ as follows (see also [10]):

$$\delta(g, h) = \inf\left\{\left(\frac{1}{2}\right)^t : t \in \mathbb{N},\ g|^t = h|^t\right\}, \quad g, h \in Aut\left(X^*\right),$$

where $g|^t := g|_{X^{(t)}}$ denotes the restriction of g to the set $X^{(t)}$. The mapping $g \mapsto (g|^i)_{i\in\mathbb{N}}$, $g \in Aut(X^*)$ identifies the group $Aut(X^*)$ with the infinitely iterated permutational wreath product $\wr_{i=1}^{\infty} Sym(X_i)$ of symmetric groups on the consecutive alphabets.

The above geometric interpretation allows to identify the infinitely iterated wreath product (1) with a closed subgroup of $Aut(X^*)$ consisting of all automorphisms g such that for every word $w \in X^*$ the restriction of g to the set $wX_{|w|+1}$ (the so-called vertex permutation of g at the vertex w, denoted further by $g[w]$) belongs to the group $G_{|w|+1}$, where $|w|$ denotes the length of the word w (see Section 4.4).

Profinite groups of the form W_∞ play a particular role not only in the group theory, where, for example, they describe profinite completions of some branch groups [13] and provide interesting examples and counterexamples in the theory of profinite groups (see, for example, [4,21,25,26]). Iterated permutational wreath products of finite groups arise also as symmetry groups of such combinatorial structures as nested designs (see [1,2]) or in chemistry they describe symmetries of certain non-rigid molecules (see [3,29]). Nowadays, these groups are even found to be useful as descriptors for processing information in the human visual system [20]. Note that the study of iterated wreath products was initiated by L. Kaloujnine [16,17] in the mid-40s last century. This was continued by his students I.D. Ivanyuta [15], V.I. Sushchansky [7,27,28], Y.V. Bodnarchuk [7] and others.