Journal of Pure and Applied Algebra 219 (2015) 1639–1644

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Isomorphism of finitely generated solvable groups is weakly universal ☆

Jay Williams [1]

Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA

ARTICLE INFO

Article history:
Received 18 April 2014
Available online 10 July 2014
Communicated by C.A. Weibel

ABSTRACT

We show that the isomorphism relation for finitely generated solvable groups of class 3 is a weakly universal countable Borel equivalence relation. This improves on previous results. The proof uses a modification of a construction of Neumann and Neumann. Elementary arguments show that isomorphism of finitely generated metabelian or nilpotent groups can not achieve this Borel complexity. In this sense the result is sharp, though it remains open whether the relation is in fact universal.

1. Introduction

Let E be an equivalence relation defined on a Polish space X. Then we call E *Borel* if E is Borel as a subset of X^2. We say E is *countable* if every equivalence class of E is countable. Given two equivalence relations E, F on Polish spaces X, Y respectively, we say that E *Borel reduces* to F, written $E \leq_B F$, if there is a Borel map $f: X \to Y$ such that

$$x \, E \, y \quad \Leftrightarrow \quad f(x) \, F \, f(y)$$

Intuitively, if $E \leq_B F$, then the classification problem associated to F is at least as difficult as the classification problem associated to E. Thus the notion of a Borel reduction gives us a mathematical framework for comparing the complexity of assorted classification problems. In the particular case of countable Borel equivalence relations, the following definition is especially important.

Definition 1.1. Suppose that E is a countable Borel equivalence relation. Then E is universal if for every countable Borel equivalence relation F, $F \leq_B E$.

☆ This research was partially supported by NSF Grant 1044448, Collaborative Research: EMSW21-RTG: Logic in Southern California.

E-mail address: jaywill@caltech.edu.

[1] Tel.: +1 626 3954338.

http://dx.doi.org/10.1016/j.jpaa.2014.07.002
0022-4049/

A universal countable Borel equivalence relation can then be thought of as being as complicated as possible. Thomas and Velickovic showed in [11] that isomorphism on the space of finitely generated groups $\mathcal{G}$ is universal. (For a discussion of the topology on $\mathcal{G}$, see [2].) Their proof proceeds by constructing a family of groups using free products with amalgamation. Such groups are generally large from a group-theoretic standpoint; for example nearly all contain free non-abelian subgroups. It is natural to ask whether this complexity may be achieved using "small" groups, or other more natural classes of groups. Although thus far we have not been able to prove that isomorphism of any other class of finitely generated groups is universal, we have been able to prove strong lower bounds. In order to describe them we must first record one other definition.

Definition 1.2. Suppose that E is a countable Borel equivalence relation on Y. Then E is weakly universal if for every countable Borel equivalence relation F on X there is a countable-to-1 Borel map $f: X \to Y$ such that

$$x\, F\, y \quad \Rightarrow \quad f(x)\, E\, f(y).$$

Such a map is called a countable-to-1 Borel homomorphism from F to E.

As of this writing, it is open whether or not every weakly universal equivalence relation is in fact universal. Conjecturally the answer is no, but the same conjecture implies that weakly universal equivalence relations are still much more complex than non-weakly-universal equivalence relations (see [9] for details). In any event, we expect that weakly universal equivalence relations are very complex. Recently, in [12], the author and Thomas established the following.

Theorem 1.3. *(See Thomas and Williams [12].) Isomorphism of Kazhdan groups is weakly universal.*

The class of Kazhdan groups contains many groups which are large or complex from a group-theoretic perspective; for example there are SQ-universal Kazhdan groups. One might ask if the groups themselves must be complex if isomorphism restricted to those groups is to be universal. As there are only countably many finitely generated abelian, nilpotent, polycyclic, and metabelian groups, isomorphism for these groups is simple with respect to Borel reducibility, and so we must look to more complicated groups. The main result of this paper is the following.

Theorem 1.4. *Isomorphism of finitely generated solvable groups of class 3 is weakly universal.*

Corollary 1.5. *Isomorphism of finitely generated amenable groups is weakly universal.*

Thus isomorphism of finitely generated groups is weakly universal on a class of small groups, and in fact for the smallest solvability class possible. Further, this improves the previous best known lower bound on the complexity of isomorphism of finitely generated amenable groups, which was established by Thomas in [10], based on the work of Giordano, Putnam and Skau [4], Bezuglyi and Medynets [1], Matui [7], and Juschenko and Monod [5] on topological full groups. The proof of Theorem 1.4 is based on a construction due to Neumann and Neumann in [8] and uses nothing more complicated than wreath products. There are other constructions which show that solvable groups can be quite complicated from an algorithmic point of view, such as the one from [6], so the result is not entirely unexpected. It does not finish the story, however.

Conjecture. *Isomorphism of finitely generated solvable groups of class 3 is a universal countable Borel equivalence relation.*