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On excess in finite Coxeter groups ☆

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ABSTRACT

For a finite Coxeter group W and w an element of W the *excess* of w is defined to be $e(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy,\ x^2 = y^2 = 1\}$ where ℓ is the length function on W. Here we investigate the behaviour of $e(w)$, and a related concept reflection excess, when restricted to standard parabolic subgroups of W. Also the set of involutions inverting w is studied.

1. Introduction

This paper, continuing the investigations begun in [6] and [7], studies further properties of excess in Coxeter groups. First we recall the definition of excess.

Suppose W is a Coxeter group with length function ℓ, and set

$$\mathcal{W} = \{w \in W \mid w = xy \text{ where } x, y \in W \text{ and } x^2 = y^2 = 1\}.$$

Then for $w \in \mathcal{W}$, the excess of w is

$$e(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy,\ x^2 = y^2 = 1\}.$$

The length function is not additive in general. The relationship between $\ell(w_1 w_2)$ and $\ell(w_1) + \ell(w_2)$ for various special cases of $w_1, w_2 \in W$ appears in several well-known results. For example, it is an important fact that if W_J is a standard parabolic subgroup of W, then there is a set X_J of so-called distinguished right coset representatives for W_J in W with the property that $\ell(wx) = \ell(w) + \ell(x)$ for all $w \in W_J$, $x \in X_J$

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[9, Proposition 1.10]. There is a parallel statement to this for double cosets of two standard parabolic subgroups of W [5, Proposition 2.1.7]. Also, when W is finite it possesses an element w_0, the longest element of W, for which $\ell(w_0) = \ell(w) + \ell(ww_0)$ for all $w \in W$ [5, Lemma 1.5.3]. Another special feature of finite Coxeter groups is that every element can be written (in possibly many ways) as a product xy where $x^2 = y^2 = 1$, and it seems natural to ask the extent to which additivity of length, as measured by excess, is achieved. The hope is that investigation into excess will yield useful additions to the techniques available for the study of Coxeter groups.

The main result in [6] asserts that every element in $\mathcal{W}$ is W-conjugate to an element whose excess is zero (see also [8]). In a similar vein, [7] shows that if W is a finite Coxeter group, then every W-conjugacy class possesses at least one element which simultaneously has minimal length in the conjugacy class and excess equal to zero. The present paper explores other properties of excess in finite Coxeter groups. So from now on we assume W is finite, which implies that $\mathcal{W} = W$. Moreover, every element $w \in W$ may be written as xy, where $x^2 = y^2 = 1$ and $L(w) = L(x) + L(y)$, where L is the reflection length function on W. This fact is established for finite Weyl groups in Carter [3] (and easily verified for the remaining finite Coxeter groups). This leads to the related notion of reflection excess. For $w \in W$ its reflection excess $E(w)$ is defined by

$$E(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy,\ x^2 = y^2 = 1,\ L(w) = L(x) + L(y)\}.$$

Clearly $E(w) \geq e(w)$. However $E(w)$ and $e(w)$ can be markedly different – see for example Proposition 3.3 of [7].

The first issue we address here is how excess and reflection excess behave on restriction to standard parabolic subgroups of W – as is well-known, such subgroups are Coxeter groups in their own right. If W_J is a standard parabolic subgroup of W and $w \in W_J$, we let $e_J(w)$ (respectively $E_J(w)$) be the excess of w (respectively reflection excess of w) considered as an element of W_J.

Our main results are as follows.

Theorem 1.1. *Let W_J be a standard parabolic subgroup of W and let $w \in W_J$. Then $E_J(w) = E(w)$.*

We remark that the proof of Theorem 1.1 is very short and elementary, whereas its sister statement for excess requires a lengthy case-by-case analysis. More than that there is a shock in store as we now see.

Theorem 1.2. *Let W_J be a standard parabolic subgroup of W and let $w \in W_J$. If W has no irreducible factors of type D_n, then $e_J(w) = e(w)$.*

The assumption that there be no direct factors of type D_n in Theorem 1.2 cannot be omitted. In Section 3 we give an example with W of type D_{12}, W_J of type D_{11} and an element w of W_J for which $e_J(w) = 60$ but $e(w) = 46$. However there are a number of positive results, to be found in Section 3, for W of type D_n provided we restrict W_J.

For $w \in W$, the set $\mathcal{I}_w$, which is defined as follows,

$$\mathcal{I}_w = \{x \in W \mid x^2 = 1,\ w^x = w^{-1}\}$$

is intimately connected with $e(w)$ and $E(w)$. This is because if x and y are elements of W, with $x^2 = y^2 = 1$, such that $xy = w$, then $w^x = (xy)^x = yx = w^{-1}$ and similarly $w^y = w^{-1}$. Therefore $x, y \in \mathcal{I}_w$. Thus $\mathcal{I}_w$ is always non-empty.

For $X \subseteq W$, we define in Section 2 a certain subset $N(X)$ of the positive roots of W. The Coxeter length (or just length) of X, denoted by $\ell(X)$, is defined to be $|N(X)|$ (see [10]). A consequence of our next theorem is that for all $w \in W$, $\ell(w) \leq \ell(\mathcal{I}_w)$.