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The second Hilbert coefficients and the homological torsions of parameters

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A B S T R A C T

Let M be a finitely generated module over a Noetherian local ring. This paper gives, for a given parameter ideal Q for M, bounds for the second Hilbert coefficients $\mathrm{e}_Q^2(M)$ in terms of the homological degrees and torsions of modules. We also report a criterion for a certain equality of the second Hilbert coefficients of parameters and the homological torsions of modules.

1. Introduction

The purpose of our paper is to study the second Hilbert coefficients of parameters in terms of the homological degrees and torsions of modules.

To state the problems and the results of our paper, first of all, let us fix some of our notation. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let M be a finitely generated A-module with $r = \dim_A M$. For simplicity, throughout this paper, we assume that A is $\mathfrak{m}$-adically complete and the residue class field $A/\mathfrak{m}$ of A is infinite.

For each $j \in \mathbb{Z}$ we set

$$M_j = \mathrm{Hom}_A\big(\mathrm{H}^j_{\mathfrak{m}}(M), E\big),$$

where $E = \mathrm{E}_A(A/\mathfrak{m})$ denotes the injective envelope of $A/\mathfrak{m}$ and $\mathrm{H}^j_{\mathfrak{m}}(M)$ the jth local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$. Then M_j is a finitely generated A-module with $\dim_A M_j \le j$ for all $j \in \mathbb{Z}$ [6, Fact 2.1]. Let I be a fixed $\mathfrak{m}$-primary ideal in A and let $\ell_A(N)$ denote, for an A-module N, the length of N. Then there exist integers $\{\mathrm{e}^i_I(M)\}_{0 \le i \le r}$ such that

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$$\ell_A\big(M/I^{n+1}M\big) = \mathrm{e}_I^0(M)\binom{n+r}{r} - \mathrm{e}_I^1(M)\binom{n+r-1}{r-1} + \cdots + (-1)^r \mathrm{e}_I^r(M)$$

for all $n \gg 0$. We call $\mathrm{e}_I^i(M)$ the ith Hilbert coefficient of M with respect to I and especially call the leading coefficient $\mathrm{e}_I^0(M)$ (> 0) the multiplicity of M with respect to I.

The homological degree $\mathrm{hdeg}_I(M)$ of M with respect to I is inductively defined in the following way, according to the dimension $r = \dim_A M$ of M.

Definition 1.1. (See [13].) For each finitely generated A-module M with $r = \dim_A M$, we set

$$\mathrm{hdeg}_I(M) = \begin{cases} \ell_A(M) & \text{if } r \le 0, \\ \mathrm{e}_I^0(M) + \sum_{j=0}^{r-1} \binom{r-1}{j} \mathrm{hdeg}_I(M_j) & \text{if } r > 0 \end{cases}$$

and call it the homological degree of M with respect to I.

Then the homological torsions of modules are defined as follows.

Definition 1.2. Let M be a finitely generated A-module with $r = \dim_A M \ge 2$. We set

$$\mathrm{T}_I^i(M) = \sum_{j=1}^{r-i} \binom{r-i-1}{j-1} \mathrm{hdeg}_I(M_j)$$

for each $1 \le i \le r-1$ and call them the homological torsions of M with respect to I.

Notice that the homological degrees $\mathrm{hdeg}_I(M)$ and torsions $\mathrm{T}_I^i(M)$ of M with respect to I depend only on the integral closure of I.

In [3], it was proved that for parameter ideals Q for M, there is a lower bound

$$\mathrm{e}_Q^1(M) \ge -\mathrm{T}_Q^1(M)$$

of the first Hilbert coefficient $\mathrm{e}_Q^1(M)$ in terms of the homological torsion $\mathrm{T}_Q^1(M)$. Recently, in [6, Theorem 1.4], the authors showed that the equality $\mathrm{e}_Q^1(M) = -\mathrm{T}_Q^1(M)$ holds true if and only if the equality $\chi_1(Q;M) = \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M)$ holds true for parameter ideals Q for an unmixed module M, where $\chi_1(Q;M) = \ell_A(M/QM) - \mathrm{e}_Q^0(M)$ denotes the first Euler characteristic of M relative to Q. Here we notice that the inequality $\mathrm{e}_Q^1(M) \le 0$ holds true for every parameter ideal Q of M [9, Theorem 3.5] and that M is a Cohen–Macaulay A-module once $\mathrm{e}_Q^1(M) = 0$ for some parameter ideal Q, provided M is unmixed (see [2,3]). Thus the behavior of the first Hilbert coefficients $\mathrm{e}_Q^1(M)$ for parameter ideals Q for M is rather satisfactorily understood.

The purpose of this paper is to study the natural question of how about the second Hilbert coefficients $\mathrm{e}_Q^2(M)$ of M with respect to Q. For the estimation of $\mathrm{e}_Q^2(M)$ the key is the inequalities

$$-\ell_A\big(\mathrm{H}_{\mathfrak{m}}^1(M)\big) \le \mathrm{e}_Q^2(M) \le 0$$

of parameters Q in the case where $\dim_A M = \dim A = 2$ and $\mathrm{depth}_A M > 0$ (Proposition 3.1). We will also show that $\mathrm{e}_Q^2(M) = 0$ if and only if the ideal Q is generated by a system a_1, a_2 of parameters which forms a d-sequence on M.

Then the first main result of this paper answers the question and is stated as follows. Recall that M is said to be unmixed, if $\dim A/\mathfrak{p} = \dim_A M$ for all $\mathfrak{p} \in \mathrm{Ass}_A M$ (since A is assumed to be $\mathfrak{m}$-adically complete). We also notice here that, passing to $A/\mathrm{Ann}_A(M)$, we may assume that $\dim_A M = d = \dim A$ in our theorems, where $\mathrm{Ann}_A(M)$ denotes the annihilator of M.