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Cyclic covering of the projective line with prime gonality

Nan Wangyu

Department of Mathematics, Graduate School of Science and Engineering, Saitama University, Shimo-Okubo 255, Sakura-ku, Saitama-shi, 338-8570, Japan

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ABSTRACT

Let V be a cyclic covering of the complex projective line, g be the genus of V, $\mathrm{Gon}(V)$ be the gonality of V, and p be a prime number. In this note, we prove a necessary and sufficient condition for $\mathrm{Gon}(V) = p$ when $g > (p-1)^2$. We also give a necessary and sufficient condition without the bound of g for the case where $p = 3$, and the similar result for the case where $p = 2$ has been given in the previous notes [3] and [4].

1. Introduction

Let V be a smooth connected complete curve over $\mathbb{C}$ which admits a structure of d-cyclic covering of the projective line $\mathbb{P}^1$ with n branch points. In this note, we denote the gonality of V by $\mathrm{Gon}(V)$. The condition for $\mathrm{Gon}(V) = 1$ is trivial, and the condition for $\mathrm{Gon}(V) = 2$ has been given in [3] and [4]. Thus, we may always assume that $\mathrm{Gon}(V) \geq 3$ in this note. In particular, we can let $d \geq 3$ and $n \geq 3$, so V has the following plane model:

$$\begin{aligned} V\colon \quad & y^d = (x-\lambda_1)^{a_1}(x-\lambda_2)^{a_2}\cdots(x-\lambda_n)^{a_n}, \quad a_i \not\equiv 0 \pmod{d}, \\ & d, n \geq 3, \quad \gcd(d, a_1, \ldots, a_n) = 1, \quad a_1 + \cdots + a_n \equiv 0 \pmod{d}, \end{aligned} \tag{1.1}$$

where the branch points $\lambda_i \in \mathbb{P}^1$ are mutually distinct with each other. If ρ is a projective transformation of $\mathbb{P}^1$, and $\lambda_i' = \rho(\lambda_i)$, then V is birational to the curve

$$y^d = \left(x-\lambda_1'\right)^{a_1}\left(x-\lambda_2'\right)^{a_2}\cdots\left(x-\lambda_n'\right)^{a_n}.$$

If some λ_i, say λ_n, is just taken to the infinite point of $\mathbb{P}^1$, then the equation of V should be written in the form

E-mail address: nanwang0824@gmail.com.

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$$y^d = (x-\lambda_1)^{a_1}(x-\lambda_2)^{a_2}\cdots(x-\lambda_{n-1})^{a_{n-1}},$$

and in this case, the number a_n is recovered as

$$a_n = \min\{a' \geq 1 : a_1 + \cdots + a_{n-1} + a' \equiv 0 \pmod{d}\}.$$

Definition 1. We call $(d; a_1, \ldots, a_n)$ the *type* of the curve V given in (1.1). For fixed d and n, two types $(d; a_1, \ldots, a_n)$ and $(d; a'_1, \ldots, a'_n)$ are said to be in the same *Nielsen class* if $(a'_1, \ldots, a'_n) \equiv (ka_{\tau(1)}, \ldots, ka_{\tau(n)})$ (mod d) for some permutation $\tau \in \mathcal{S}_n$ and some integer k relatively prime to d. In this case, we write

$$(a'_1, \ldots, a'_n) \equiv^k_\tau (a_1, \ldots, a_n) \pmod{d}.$$

Definition 2. We write $\mathbb{Z}_d^\times = \{k \in \mathbb{Z} : 1 \leq k \leq d-1,\ \gcd(d,k) = 1\}$. Let V be the curve given in (1.1). For every $k \in \mathbb{Z}_d^\times$, there exists a permutation $\tau \in \mathcal{S}_n$ and integers $\{a'_i\}_{i=1}^n$ such that

$$(a'_1, \ldots, a'_n) \equiv^k_\tau (a_1, \ldots, a_n) \pmod{d} \quad \text{and} \quad 1 \leq a'_1 \leq \cdots \leq a'_n \leq d-1.$$

Then let $n_k = (\sum_{i=1}^n a'_i)/d$ and $r_k = \sum_{i=1}^{n-n_k} a'_i$, and we define

$$r_{\min}(d; a_1, \ldots, a_n) = \min\{r_k : k \in \mathbb{Z}_k^\times\}.$$

Theorem 3. *Let V be the curve given in (1.1), g be its genus, and p be an odd prime number. When $g > (p-1)^2$, we have* $\mathrm{Gon}(V) = p$ *if and only if one of the following conditions holds:*

(G1) *d is divisible by p, and at least $n-2$ of the a_i's are divisible by d/p;*
(G2) *the condition* (G1) *does not hold, and $r_{\min}(d; a_1, \ldots, a_n) = p$.*

Corollary 4. *When $g > (p-1)^2$, whether the gonality of curve V given in (1.1) equals the prime number p is determined by the Nielsen class of the type of V, and independent of the choice of the parameters λ_i's.*

Remark 5. The condition (G1) is quite easy to understand, and we consider the condition (G2). Let the integers d, n and p be fixed, and assume that $g > (p-1)^2$. Then we see that $r_{\min}(d; a_1, \ldots, a_n) = p$ if and only if there exists a permutation $\tau \in \mathcal{S}_n$, an integer $k \in \mathbb{Z}_d^\times$ and positive integers s, t, $u_1, \ldots, u_s$, $v_1, \ldots, v_t$ such that the following conditions hold:

(1) $s + t = n$,
(2) $p = u_1 + \cdots + u_s = v_1 + \cdots + v_t$,
(3) $\max\{u_i\}_{i=1}^s + \max\{v_j\}_{j=1}^t \leq d$,
(4) $(d; a_1, \ldots, a_n) \equiv^k_\tau (d; u_1, \ldots, u_s, d-v_1, \ldots, d-v_t) \pmod{d}$.

Hence, we can easily compute and list all the Nielsen classes whose corresponding curves satisfy the condition (G2). (See the computation for $p = 3$ in Section 3 as an example.) Furthermore, by Proposition 8 latter, the condition (G2) implies that the curve V can be birationally transformed to the following form:

$$V': \quad y^d = (x-\mu_1)^{u_1}\cdots(x-\mu_s)^{u_s} \cdot (x-\mu_{s+1})^{d-v_1}\cdots(x-\mu_n)^{d-v_t},$$

and the branch points μ_i's can be taken away from the infinite point of $\mathbb{P}^1$. In this case, we have $\deg(y/(x-\mu_{s+1})\cdots(x-\mu_n)) = p$ on V'.