Journal of Pure and Applied Algebra 219 (2015) 1742–1760

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

On mutations of selfinjective quivers with potential

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ARTICLE INFO

Article history:
Received 17 October 2012
Received in revised form 7 June 2014
Available online 8 July 2014
Communicated by S. Koenig

ABSTRACT

We study silting mutations (Okuyama–Rickard complexes) for selfinjective algebras given by quivers with potential (QPs). We show that silting mutation is compatible with QP mutation. As an application, we get a family of derived equivalences of Jacobian algebras.

1. Introduction

Derived categories are nowadays considered as an essential tool in the study of many areas of mathematics. In the representation theory of algebras, derived equivalences of algebras have been one of the central themes and extensively investigated. It is well-known that endomorphism algebras of tilting complexes are derived equivalent to the original algebra [25]. Therefore it is an important problem to give concrete methods to calculate endomorphism algebras of tilting complexes. In this paper, we focus on one of the fundamental tilting complexes over selfinjective algebras, known as Okuyama–Rickard complexes, which play an important role in the study of Broué's abelian defect group conjecture. From a categorical viewpoint, they are nowadays interpreted as a special case of silting mutation [2]. We provide a method to determine the quivers with relations of the endomorphism algebras of Okuyama–Rickard complexes when selfinjective algebras are given by quivers with potential (QPs for short).

The notion of QPs was introduced by [11], which gives a better understanding of cluster algebras (we refer to [18]). Recently it has been discovered that mutations of QPs (Definition 2.2) give rise to derived equivalences in several situations, for example [8,16,20,22–24,26]. In particular, the deep connection between mutations and derived equivalences is quite useful to study the derived equivalence classification of cluster-tilted algebras [5–7,10]. The aim of this paper is to give a similar (but different) type of derived equivalences by comparing QP mutation and silting mutation (Definition 2.4).

Our main result is the following (see Sections 2 and 3 for unexplained notions).

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http://dx.doi.org/10.1016/j.jpaa.2014.07.010

Theorem 1.1 *(Proposition 2.7, Theorem 3.1, Corollary 3.2 and Lemma 3.4). Let (Q, W) be a selfinjective QP (Definition 2.1) and $\Lambda := \mathcal{P}(Q, W)$. For a set of vertices $I \subset Q_0$, we assume the following conditions.*

- *Any vertex in I is not contained in 2-cycles in Q.*
- *There are no arrows between vertices in I.*

(a) *We have an algebra isomorphism*

$$\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)}\big(\mu_I(\Lambda)\big) \cong \mathcal{P}\big(\mu_I(Q, W)\big),$$

where $\mu_I(\Lambda)$ is left (or right) silting mutation and $\mu_I(Q, W)$ is a composition of QP mutation of the vertices I.

(b) *If $\sigma I = I$ for the Nakayama permutation σ of Λ, then $\mu_I(\Lambda)$ is a tilting complex. In particular, Λ and $\mathcal{P}(\mu_I(Q, W))$ are derived equivalent.*

Since selfinjective algebras are closed under derived equivalence, we conclude that from (b) above the new QP is also a selfinjective QP, which is a result given in [15, Theorem 4.2]. Then we can apply our result to the new QP again and these processes provide a family of derived equivalences. We note that Keller and Yang [20] proved that, for two QPs related by QP mutation, their Ginzburg dg algebras, which are certain enhancement of Jacobian algebras, are derived equivalent though their Jacobian algebras are far from being derived equivalent in general. On the other hand, Theorem 1.1 tells us that Jacobian algebras are already derived equivalent in our setting.

Notations Let K be an algebraically closed field and $D := \mathrm{Hom}_K(-, K)$. All modules are left modules. For a finite dimensional algebra Λ, we denote by $\mathrm{mod}\,\Lambda$ the category of finitely generated Λ-modules and by $\mathrm{add}\,M$ the subcategory of $\mathrm{mod}\,\Lambda$ consisting of direct summands of finite direct sums of copies of $M \in \mathrm{mod}\,\Lambda$. The composition fg means first f, then g. For a quiver Q, we denote by Q_0 vertices and by Q_1 arrows of Q. We denote by $s(a)$ the start vertex and by $e(a)$ the end vertex of an arrow or path a.

2. Preliminaries

2.1. Quivers with potential

We recall the definition of quivers with potential. We follow [11].

- Let Q be a finite connected quiver without loops. We denote by KQ_i the K-vector space with basis consisting of paths of length i in Q, and by $KQ_{i,\mathrm{cyc}}$ the subspace of KQ_i spanned by all cycles. We denote the *complete path algebra* by

$$\widehat{KQ} = \prod_{i \geq 0} KQ_i$$

and by $J_{\widehat{KQ}}$ the Jacobson radical of $\widehat{KQ}$. A *quiver with potential* (QP) is a pair (Q, W) consisting of a finite connected quiver Q without loops and an element $W \in \prod_{i \geq 2} KQ_{i,\mathrm{cyc}}$, called a *potential*. For each arrow a in Q, the *cyclic derivative* $\partial_a : \widehat{KQ}_{\mathrm{cyc}} \to \widehat{KQ}$ is defined as the continuous linear map satisfying $\partial_a(a_1 \cdots a_d) = \sum_{a_i = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}$ for a cycle $a_1 \cdots a_d$. For a QP (Q, W), we define the *Jacobian algebra* by

$$\mathcal{P}(Q, W) = \widehat{KQ}/\mathcal{J}(W),$$