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Tilting bundles and the missing part on a weighted projective line of type $(2, 2, n)$ ☆

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ABSTRACT

This paper is devoted to investigate the tilting bundles, i.e. tilting sheaves that are vector bundles, in the category of coherent sheaves on a weighted projective line of type $(2, 2, n)$. We classify all the tilting bundles into two classes, one containing the tilting bundles that are consisting of line bundles, and the other one containing tilting bundles with indecomposable summands that are not line bundles. Moreover, for each tilting bundle T (with endomorphism algebra Λ) we prove that the missing part, from the category of coherent sheaves to the category of finitely generated right Λ-modules, carries the structure of an abelian category.

1. Introduction

Tilting theory arises from the representation theory of finite dimensional algebras and has been proved to be a universal method for constructing equivalences between categories (see for instance [1,5]). Geigle and Lenzing [4] extended the notion of tilting modules to tilting sheaves in the category $\mathrm{coh}\,\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$. Parallel to the situation in module categories, they showed that each tilting sheaf T (with endomorphism algebra Λ) gives rise to a torsion pair $(\mathcal{X}_0, \mathcal{X}_1)$ in $\mathrm{coh}\,\mathbb{X}$ and a torsion pair $(\mathcal{Y}_1, \mathcal{Y}_0)$ in the category $\mathrm{mod}(\Lambda)$ of finitely generated right Λ-modules by setting

$$\mathcal{X}_i = \left\{ X \in \mathrm{coh}\,\mathbb{X} \;\middle|\; \mathrm{Ext}^j_{\mathrm{coh}\,\mathbb{X}}(T, X) = 0,\ j \neq i \right\}$$

and

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$$\mathcal{Y}_i = \{ Y \in \mathrm{mod}(\Lambda) \mid \mathrm{Tor}_j^{\Lambda}(Y,T) = 0,\ j \neq i \}$$

for $i = 0, 1$. Moreover, the functors

$$\mathrm{Ext}^i_{\mathrm{coh}\,\mathbb{X}}(T,-) : \mathcal{X}_i \to \mathcal{Y}_i \quad \text{and} \quad \mathrm{Tor}_i^{\Lambda}(-,T) : \mathcal{Y}_i \to \mathcal{X}_i$$

define equivalences of categories, mutually inverse to each other, for $i = 0, 1$. In particular, the torsion pair $(\mathcal{Y}_1, \mathcal{Y}_0)$ is splitting.

Lenzing [8] pointed out that if the endomorphism algebra Λ is not hereditary, there must miss some objects from $\mathrm{coh}\,\mathbb{X}$ to $\mathrm{mod}(\Lambda)$. For the reasons that the torsion pair $(\mathcal{Y}_1, \mathcal{Y}_0)$ is splitting in $\mathrm{mod}(\Lambda)$, and that there are equivalences between $\mathcal{X}_i$ and $\mathcal{Y}_i$ for $i = 0, 1$, it is a natural way to describe the missing part from $\mathrm{coh}\,\mathbb{X}$ to $\mathrm{mod}(\Lambda)$ as the factor category $\mathscr{C} := \mathrm{coh}\,\mathbb{X}/[\mathcal{X}_0 \cup \mathcal{X}_1]$, where $[\mathcal{X}_0 \cup \mathcal{X}_1]$ denotes the two-sided ideal of all morphisms in $\mathrm{coh}\,\mathbb{X}$ which factor through a finite direct sum of coherent sheaves in $\mathcal{X}_0 \cup \mathcal{X}_1$.

It is natural to ask what structure the missing part may carry. In general, the factor category $\mathscr{C}$ is only an additive category. In [3], we focused on the weighted projective lines of type $(2, 2, n)$ to show that for canonical tilting sheaf $T_{\mathrm{can}} = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$, the corresponding missing part is an abelian category which is equivalent to the module category $\mathrm{mod}(k\vec{\mathbb{A}}_{n-1})$, where $k\vec{\mathbb{A}}_{n-1}$ denotes the path algebra of the equioriented quiver $\mathbb{A}_{n-1}$. Moreover, several examples there indicate that the missing part is not, in general, abelian if the weight type is different from $(2, 2, n)$, or if the tilting sheaf contains a direct summand of finite length. In this paper, we extend the main result of [3] to a more general case. Namely, we show that for weight type $(2, 2, n)$, the missing part induced by any tilting bundle carries the structure of an abelian category.

The paper is organized as follows. In Section 2, we recall some basic definitions and well-known results on the category of coherent sheaves on a weighted projective line. In Section 3, we classify all the tilting bundles into two classes, one containing the tilting bundles that are consisting of line bundles, and the other one containing tilting bundles with indecomposable summands that are not line bundles. For the former case, each tilting bundle can be obtained from the canonical tilting bundle T_{can} by a line bundle twist; for the latter case, we give the trichotomy of the form for each tilting bundle, that is, such a tilting bundle T can be decomposed into three parts, $T = T^+(E_i) \oplus (\bigoplus_{i \leq k \leq j} E_k) \oplus T^-(E_j)$, see (3.16) for details. In Section 4, we show that for any tilting bundle, the corresponding missing part is an abelian category. More precisely, the missing part is equivalent to a module category over a path algebra of type A for the tilting bundles of the former case, and it is equivalent to a product of two such categories for the tilting bundles of the latter case.

To simplify notation, we denote $\mathrm{Ext}^i_{\mathrm{coh}\,\mathbb{X}}(-,-)$ by $\mathrm{Ext}^i(-,-)$ for $i \geq 0$.

2. Preliminary

The notion of weighted projective lines was introduced by Geigle and Lenzing [4] for a geometric treatment to canonical algebras which were studied by Ringel [9]. In this section, we recall some basic definitions and well-known results on the category of coherent sheaves on a weighted projective line of type $(2, 2, n)$.

2.1. Coherent sheaves on a weighted projective line $\mathbb{X}$

Let k be an algebraically closed field, let $\mathbb{P}^1$ be the projective line over k, let $\mathbf{D} = (\infty, 0, 1)$ be a sequence of points of $\mathbb{P}^1$, and let $\mathbf{p} = (2, 2, n)$ be a sequence of integers with $n \geq 2$. In [4], the triple $\mathbb{X} = (\mathbb{P}^1, \mathbf{D}, \mathbf{p})$ is called a weighted projective line of weight type $(2, 2, n)$.

Denote by S the commutative algebra

$$S = k[X_1, X_2, X_3]/\langle X_3^n - X_2^2 + X_1^2 \rangle \triangleq k[x_1, x_2, x_3].$$