Journal of Pure and Applied Algebra 219 (2015) 2658–2665

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Graver degrees are not polynomially bounded by true circuit degrees

Christos Tatakis [a], Apostolos Thoma [b,*]

[a] *Mitilini, P.O. Box 13, Mitilini (Lesvos) 81100, Greece*
[b] *Department of Mathematics, University of Ioannina, Ioannina 45110, Greece*

ARTICLE INFO

Article history:
Received 10 July 2014
Available online 16 October 2014
Communicated by A.V. Geramita

MSC:
14M25; 05C25; 05C38

ABSTRACT

Let I_A be a toric ideal. We prove that the degrees of the elements of the Graver basis of I_A are not bounded above by a polynomial on the maximal true degree of the circuits of I_A.

1. Introduction

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subseteq \mathbb{N}^n$ be a vector configuration in $\mathbb{Q}^n$ and $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ the corresponding affine semigroup. We grade the polynomial ring $\mathbb{K}[x_1, \ldots, x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the *A-degree* of the monomial $\mathbf{x}^\mathbf{u} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A.$$

We denote it by $\deg_A(\mathbf{x}^\mathbf{u})$, while the usual degree $u_1+\cdots+u_m$ of $\mathbf{x}^\mathbf{u}$ we denote by $\deg(\mathbf{x}^\mathbf{u})$. The *toric ideal* I_A associated to A is the prime ideal generated by all the binomials $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}$ such that $\deg_A(\mathbf{x}^\mathbf{u}) = \deg_A(\mathbf{x}^\mathbf{v})$, see [5]. A nonzero binomial $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}$ in I_A is called *primitive* if there exists no other binomial $\mathbf{x}^\mathbf{w} - \mathbf{x}^\mathbf{z}$ in I_A such that $\mathbf{x}^\mathbf{w}$ divides $\mathbf{x}^\mathbf{u}$ and $\mathbf{x}^\mathbf{z}$ divides $\mathbf{x}^\mathbf{v}$. The set of the primitive binomials forms the Graver basis of I_A and is denoted by Gr_A. An irreducible binomial is called a *circuit* if it has minimal support, where the support of a binomial $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}$ is the set of all $\mathbf{a}_i$ such that $x_i|\mathbf{x}^\mathbf{u}$ or $x_i|\mathbf{x}^\mathbf{v}$. The set of the circuits is denoted by $\mathcal{C}_A$ and it is a subset of the Graver basis, see [5]. The Graver basis contains also every minimal system of generators of I_A and the universal Gröbner basis, the union of all reduced Gröbner bases, see [5].

* Corresponding author.
E-mail addresses: chtataki@cc.uoi.gr (C. Tatakis), athoma@uoi.gr (A. Thoma).

http://dx.doi.org/10.1016/j.jpaa.2014.09.021
0022-4049/

One of the fundamental problems in toric algebra is to give good upper bounds on the degrees of the elements of the Graver basis. These bounds have important implications to integer programming and computational algebraic geometry, see [1,5,6]. There exist several bounds on the degrees of the elements of the Graver basis of a toric ideal, see for example [1,5,7]. It is an interesting question if the maximum degree of a circuit or the maximum *true* degree of a circuit bounds the maximum degree of any Graver basis element, see [5, Chapter 4], [6, Conjecture 4.8], [1, Conjecture 2.2.10]. Following [6] we define the true degree of a circuit as follows: Consider any circuit $C \in \mathcal{C}_A$, the lattice $\mathbb{Z}(\text{supp}(C))$ has finite index in the lattice $\mathbb{R}(\text{supp}(C)) \cap \mathbb{Z}A$, which is called the index of the circuit C and is denoted by $\text{index}(C)$. The *true degree* of the circuit C is the product $\deg(C) \cdot \text{index}(C)$. The crucial role of the true circuit degrees was first highlighted in Hosten's dissertation [1].

Let us call t_A the maximal true degree of any circuit in $\mathcal{C}_A$. The true circuit conjecture says that

$$\deg(B) \leq t_A,$$

for every $B \in Gr_A$. There are several examples of families of toric ideals where the true circuit conjecture is true, see for example [3]. It is also true for some families of toric ideals of graphs, see [7, Section 4]. However it is not true in the general case. In [7] we gave an infinite family of counterexamples to the true circuit conjecture by providing toric ideals and elements of the Graver basis for which their degrees are not bounded above by t_A. We note that in the counterexamples of [7] the degrees of the elements of the Graver basis were bounded by t_A^2. In this article we consider the following question:

Question. Does the degree of any element in the Graver basis Gr_A of a toric ideal I_A is bounded above by a constant times $(t_A)^2$ or a constant times $(t_A)^{2014}$?

To disprove such a statement, one has to produce examples of toric ideals such that there exist elements in their Graver basis of very high degree and at the same time the true degrees of their circuits have to be relatively low. In order to achieve that, one needs to compute the Graver basis and the set of circuits for toric ideals I_A in a polynomial ring with a huge number of variables. This procedure is computationally demanding, if not impossible. An alternative approach is given by the class of the toric ideals of graphs where we explicitly know the form of the elements of their Graver basis, see [4], and of their circuits, see [8].

The main result of the article is Theorem 4.5 which says that: *there is no polynomial in t_A that bounds the degree of every element in the Graver basis Gr_A of a toric ideal I_A.*

To prove the theorem we are going to construct a family of graphs G_r^n. For the toric ideals of these graphs and for a fixed n we are going to prove that there are elements in the Graver basis whose degrees are exponential on r, see Proposition 4.1, while the true degrees of their circuits are linear on r, see Theorem 3.1 and Proposition 4.3.

This work is related with the true circuit conjecture the importance of which lies in the fact that, if it was true, it provides good upper bounds for the degrees of the generators of projective toric varieties, see [6, Conjecture 4.1]. Our main result shows that there is no polynomial in t_A or in $\text{maxdeg}(\mathcal{C}_A)$ that bounds the degree of every element in the Graver basis Gr_A of a toric ideal I_A, which is also of independent interest. Finally, this work demonstrates the power of toric ideals of graphs to produce examples of toric ideals with specific properties. This is due to the diversity of graphs and the explicit knowledge of the form of the elements of the Graver basis, the Markov bases, see [4], the circuits, see [8], and the universal Gröbner basis, see [7].

2. Toric ideals of graphs

Let G be a finite simple connected graph with vertices $V(G) = \{v_1, \ldots, v_n\}$ and edges $E(G) = \{e_1, \ldots, e_m\}$. Let $\mathbb{K}[e_1, \ldots, e_m]$ be the polynomial ring in the m variables $e_1, \ldots, e_m$ over a field $\mathbb{K}$. We