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On the singularities of effective loci of line bundles

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ABSTRACT

We prove that every irreducible component of semi-regular loci of effective line bundles in the Picard scheme of a smooth projective variety has at worst rational singularities. This generalizes Kempf's result on rational singularities of W_d^0 for smooth curves. We also work out an example of such loci for a ruled surface.

1. Introduction

Fix a ground field k, which is algebraically closed and of characteristic 0. Let X be a smooth projective curve of genus g. For $r, d \geq 0$, the Brill–Noether locus is defined as

$$\operatorname{supp}\big(W_d^r(X)\big) = \big\{L \in \operatorname{Pic}(X) \mid h^0(L) \geq r+1, \deg(L) = d\big\}.$$

There has been extensive research on these loci in literature (cf. [1]). A special kind of Brill–Noether loci is $W_d^0(X)$, which is the image of the Abel–Jacobi map $\varphi : X_d \to \operatorname{Pic}^d X$, where X_d denotes the dth symmetric product of X and $\operatorname{Pic}^d X$ denotes the Picard variety of degree d line bundles on X. When $d = g-1$, W_{g-1}^0 is a theta divisor if $\operatorname{Pic}^{g-1} X$ is identified with the Jacobian of the curve.

Kempf [7] proved that $W_d^0(X)$ has only rational singularities, so in particular it is Cohen–Macaulay and normal; and for $1 \leq d \leq g-1$, the tangent cone over a point $[L] \in W_d^0(X)$ admits a rational resolution from the normal bundle of the fiber $F = \varphi^{-1}([L])$. He also computed the degree of the tangent cone, generalizing Riemann's formula on multiplicity of theta divisors. A celebrated generalization of Kempf's theorem in the case $d = g-1$ is due to Ein and Lazarsfeld [4, Theorem 1] stating that any principal polarization divisor $\Theta \subset A$ on an abelian variety is normal and has rational singularities.

This paper attempts to extend part of Kempf's results on W_d^0 for curves to higher dimensional varieties using the approach and technique of Ein [3], where the author studied the normal sheaf $\mathcal{N}$ of the fiber F. He showed $\mathcal{N}$ can be reconstructed from the multiplication map $H^0(\mathcal{O}_F(1)) \otimes H^1(\mathcal{N}(-1)) \to H^1(\mathcal{N})$, and

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proved that for a general curve X, $\mathcal{N} \simeq \rho\mathcal{O}_F \oplus (H^1(X,L) \otimes \Omega_F(1))$, where $\rho = g-(r+1)(g+r-d)$ is the Brill–Noether number. A large part of his results was built on a locally free resolution of $\mathcal{N}^*$.

Now let X be a smooth projective variety of arbitrary dimension. Let $\mathrm{Pic}(X)$ and $\mathrm{Div}(X)$ denote the Picard scheme and divisor scheme, which parameterize line bundles and effective divisors on X respectively. One still has the Abel–Jacobi map $\varphi : \mathrm{Div}(X) \to \mathrm{Pic}(X)$, where $\mathrm{Div}(X)$ plays the same role as X_d. However, as a closed subscheme of the Hilbert scheme $\mathrm{Hilb}(X)$, $\mathrm{Div}(X)$ may be very singular. Even for $\dim X = 2$, an example due to Severi and Zappa in 1940s shows that $\mathrm{Div}(X)$ can be nonreduced. For this reason, we restrict ourselves to those so-called semi-regular line bundles (see Section 2 for definition), and consider the semi-regular locus $W^0_{\mathrm{sr}}(X)$ they form in $\mathrm{Pic}(X)$. We refer to [9] for background of $\mathrm{Pic}(X)$ and $\mathrm{Div}(X)$. Our main theorem is

Theorem 1.1. *Let X be a smooth projective variety. Then any irreducible component of $W^0_{\mathrm{sr}}(X)$ has only rational singularities.*

When X is a curve, $W^0_{\mathrm{sr}} = \coprod_{d\geq 0} W^0_d$ and W^0_d is irreducible, and so the theorem recovers Kempf's result that W^0_d has rational singularities.

The paper is organized as follows: in Section 2, we study the conormal sheaf of fibers of the Abel–Jacobi map. We derive a resolution of the sheaf and obtain several interesting consequences. With a criterion of rational singularities based on Kovács's work, we prove our main theorem. In Section 3, one example of an irreducible component W^0_{sr} of a ruled surface is analyzed in detail. In Appendix A we prove an auxiliary result on varieties swept out by linear spans of divisors of linear systems on an embedded curve.

2. Rational singularities of $W^0_{\mathrm{sr}}(X)$

2.1. Semi-regular line bundles and their loci

Let X be a smooth projective variety. It is well known that $\mathrm{Pic}(X)$ is separated and smooth over k. As $\mathrm{Hilb}(X)$ breaks into connected components according to Hilbert polynomials, so does $\mathrm{Pic}(X)$. Fix an ample line bundle $\mathcal{O}_X(1)$ on X. For each line bundle L on X, there exists a $\mathbb{Q}$-coefficient polynomial P_L such that $P_L(n) = \chi(L(n))$ for $n \in \mathbb{Z}$. The P_L is constant over any connected component of $\mathrm{Pic}(X)$. The Abel–Jacobi map $\varphi : \mathrm{Div}(X) \to \mathrm{Pic}(X)$, which sends an effective divisor D to the associated line bundle $\mathcal{O}_X(D)$, is a projective morphism. For any line bundle L on X, canonically $\varphi^{-1}([L]) \simeq |L|$, where $[L]$ is the corresponding point of L in $\mathrm{Pic}(X)$ (cf. [9]).

Definition 2.1. An effective Cartier divisor D on X is *semi-regular* if the boundary map

$$\partial : H^1\big(\mathcal{O}_D(D)\big) \to H^2(\mathcal{O}_X)$$

is injective. A line bundle L is *semi-regular* if L is effective, and D is semi-regular for all $D \in |L|$.

Remark 2.2. If X is a curve, then all effective divisors, line bundles are automatically semi-regular. The reader can check that a necessary condition for L to be semi-regular is that $h^1(L) \leq q$ (see Corollary 2.13), and sufficient conditions are either $h^1(L) = 0$ or $H^1(\mathcal{O}_D(D)) = 0$ for all $D \in |L|$. The second one is however rather strong. For instance, when X is a surface and $p_g = h^0(\omega_X) > 0$, $H^1(\mathcal{O}_D(D)) = 0$ implies that $\mathrm{supp}(D) \subset \mathrm{Bs}(|\omega_X|)$.

Theorem 2.3 *(Severi–Kodaira–Spencer). Assume* $\mathrm{char}(k) = 0$. $\mathrm{Div}(X)$ *is smooth at* $[D]$ *of the expected dimension*