



Indecomposability for differential algebraic groups



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ABSTRACT

We study a notion of indecomposability in differential algebraic groups which is inspired by both model theory and differential algebra. After establishing some basic definitions and results, we prove an indecomposability theorem for differential algebraic groups. The theorem establishes a sufficient criterion for the subgroup of a differential algebraic group generated by an infinite family of subvarieties to be a differential algebraic subgroup. This theorem is used for various definability results. For instance, we show that every noncommutative almost simple differential algebraic group is perfect, solving a problem of Cassidy and Singer. We also establish numerous bounds on Kolchin polynomials, some of which seem to be of a nature not previously considered in differential algebraic geometry; in particular, we establish bounds on the Kolchin polynomial of the generators of the differential field of definition of a differential algebraic variety.

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The indecomposability theorem of Zilber, generalized to the setting of weakly categorical groups [30] the well-known theorem of algebraic group theory, which states that a family of irreducible subvarieties passing through the identity element of an algebraic group generates an algebraic subgroup [24]. The theorem is a powerful tool for definability results in groups of finite Morley rank. Zilber's theorem was generalized to the superstable (possibly infinite rank) setting by Berline and Lascar [3].

Because the theory of differentially closed fields of characteristic zero with m commuting derivations, denoted by $DCF_{0,m}$, is ω -stable, hence superstable, their generalization of Zilber's theorem holds for differential algebraic groups, which are the definable groups in $DCF_{0,m}$. However, when $m > 1$, it is difficult to show that a differential algebraic group satisfies the connectivity hypothesis of the Berline–Lascar indecomposability theorem [3]. In the case of superstable groups, the connectivity hypotheses are phrased in terms of Lascar rank. In the theory of differential algebraic groups, another notion of dimension arises more often in applications; the Kolchin polynomial is a numerical polynomial which tracks the growth of transcendence degree of a generic point of the group under application of the derivations. There is no known lower bound for Lascar rank in terms of the known differential birational invariants of the Kolchin polynomial [28]. The lack

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of control of Lascar rank in terms of the Kolchin polynomial affects both the hypothesis and conclusion of the differential algebraic case of the Berline–Lascar indecomposability theorem. With that in mind, we prove an indecomposability theorem for differential algebraic groups in which both the hypothesis and the conclusion are purely differential algebraic in nature, although the proof contains model-theoretic concepts and techniques.

Cassidy and Singer [6], introduced *almost simple* and *strongly connected* differential algebraic groups, and these notions play prominent roles here. Every differential algebraic group G has a strongly connected component H , which is a characteristic subgroup such that G/H is much smaller than G as measured by the degree of the Kolchin polynomial, which Kolchin called the *differential type* of G . In the classical theory of differential equations, this degree is the number of independent variables on which the general solution of a system of differential equations depends. A differential algebraic group is *almost simple* if every proper definable normal subgroup of G has smaller differential type than G . Almost simple differential algebraic groups are strongly connected. Using a Jordan–Hölder style theorem of [6], one sees that every differential algebraic group has a subnormal series in which the successive quotients are almost simple. The definability of the quotients follows from the fact that $DCF_{0,m}$ eliminates imaginaries. Quotient structures were also investigated by Kolchin [13]. Further, the quotients which appear are unique up to a suitable notion of isogeny. As is clear in the later sections of [6], analyzing the noncommutative almost simple groups would be aided by knowing that the groups were perfect. This follows from a corollary of our main indecomposability theorem – derived subgroups of strongly connected groups are closed in the Kolchin topology. The analysis of strongly connected groups of small typical differential dimension can be carried out [10] along the same lines as the analysis of groups of small Morley rank [7]. Perfection of the noncommutative almost simple groups also seems to be a necessary requirement for studying the groups via algebraic K -theory along the lines of [8]. We do not pursue that route in this paper; see [6] for a discussion of this topic.

We also discuss open problems which could connect our results to those of Berline and Lascar [3]. Generalizing this work to the difference-differential setting (or even more general settings) is of interest, but is not covered here. Also, though there are well-developed theories of numerical polynomials in these more general settings [14]; work along the lines of [6] (in those settings) would seem to be a prerequisite for proving results like those in this paper. The main original motivation for this work was the analysis of group theoretic properties of almost simple (and more generally strongly connected) differential algebraic groups. As mentioned above, this sort of analysis is one of the first steps in any attempt to classify almost simple differential algebraic groups. For further discussion of these issues, see [6].

In addition to the group theoretic developments in this paper, Theorem 2.13 is likely to be of independent interest in differential algebraic geometry. Start with a differential field k and a tuple $\bar{a}$ in a differential field extension.

Roughly speaking, the result says that if we extend k to a differential field K_1 in such a way that the Kolchin polynomial of $\bar{a}$ over k differs from the Kolchin polynomial of $\bar{a}$ over K_1 in a specific bounded manner, then there is an intermediate field extension, K such that the Kolchin polynomial of $\bar{a}$ over K is equal to the Kolchin polynomial of $\bar{a}$ over K_1 and K is generated as a differential field extension over k by some tuple which has a Kolchin polynomial that is similarly bounded. So, the result gives bounds on Kolchin polynomials of the generators of relative fields of definition (or simply fields of definition, assuming $k = \mathbb{Q}$). In proving Theorem 2.13, we found need for the analogue of the Lascar inequality for Kolchin polynomials. Let $A \subseteq K$. We remind the reader that the Lascar inequality says:

$$RU(\bar{a}/A\langle b \rangle) + RU(\bar{b}/A) \leq RU((\bar{a}, \bar{b})/A) \leq RU(\bar{a}/A\langle b \rangle) \oplus RU(\bar{b}/A).$$

The sum of two ordinals, denoted by $\alpha + \beta$, is the order type of $\alpha \cup \beta$ with the order given by letting every element of α be less than any element of β . We should mention that every ordinal may be written uniquely in the form

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