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Combinatorial structure of type dependency ☆

Richard Garner

Department of Mathematics, Macquarie University, NSW 2109, Australia

A R T I C L E I N F O

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A B S T R A C T

We give an account of the basic combinatorial structure underlying the notion of type dependency. We do so by considering the category of generalised algebraic theories in the sense of Cartmell, and exhibiting it as the category of algebras for a monad on a presheaf category. The objects of the presheaf category encode the basic judgements of a dependent sequent calculus, while the action of the monad encodes the deduction rules; so by giving an explicit description of the monad, we obtain an explicit account of the combinatorics of type dependency. We find that this combinatorics is controlled by a particular kind of decorated ordered tree, familiar from computer science and from innocent game semantics. Furthermore, we find that the monad at issue is of a particularly well-behaved kind: it is local right adjoint in the sense of Street–Weber. In future work, we will use this fact to describe nerves for dependent type theories, and to study the coherence problem for dependent type theory using the tools of two-dimensional monad theory.

1. Introduction

There has been much recent interest in Martin–Löf's type theory, spurred on by the discovery of remarkable links to algebraic topology and the theory of $(\infty, 1)$-categories. *Homotopy type theory* [23] extends Martin–Löf type theory with Voevodsky's *univalence axiom* and a new collection of type-formers, the *higher inductive types*; the resultant system is capable of deriving key results of homotopy theory—such as calculations of homotopy groups of spheres—in a synthetic, axiomatic manner. Models of the axioms include not only classical homotopy theory, but also "non-standard homotopy theories" described by $(\infty, 1)$-toposes (homotopical analogues of categories of sheaves); in fact, it is believed that we can view homotopy type theory as providing an internal language for $(\infty, 1)$-toposes, just as first-order geometric logic does for Grothendieck toposes.

The suitability of Martin–Löf type theory as a language for abstract homotopy theory is due to the presence of *identity types* which can be validly interpreted by the homotopy relation. The existence of identity

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E-mail address: richard.garner@mq.edu.au.

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types relies in turn on the possibility of *type dependency*: type families indexed by elements of other types. While the intuitive meaning of type dependency is clear, its syntactic expression is rather involved; a desire to understand its mathematical essence has led diverse authors [5,7,8,10,13,14,20] to describe notions of *categorical model* for dependent type theory which abstract away from the complexities of the syntax.

One aspect that remains implicit in both the syntactic and the categorical accounts is the combinatorial structure of type dependency: the structure imposed on the judgements of a dependent sequent calculus by the basic rules of weakening, projection and substitution. On the syntactic side, this combinatorics is hidden in the recursive clauses which generate the calculus; while on the categorical side, the essential role it plays in constructing models from the syntax is no longer visible in the finished product. In short, the syntactic approach fails to detect this structure by being insufficiently abstract, while the categorical approach fails to see it by being too abstract.

The objective of this paper is to elucidate the combinatorics of type dependency by adopting a viewpoint which is intermediate between the concrete syntactic one and the fully abstract categorical one. We will model dependent sequent calculi as algebras for a monad on a presheaf category (we assume the reader is familiar with the basic concepts of category theory as set out in [19]). Objects of the presheaf category will encode the basic judgement-forms of a sequent calculus; the algebraic structure imposed on them by the monad will encode the deduction rules. Now the combinatorial structure we wish to describe inheres in the action of the monad, and so by giving an explicit description of this action, we obtain an explicit account of the structure. More precisely, the underlying endofunctor of the monad describes the derivable judgements of a freely-generated sequent calculus; while the monad multiplication encodes the process of proof-tree normalisation by which such derivations are combined.

For the dependent sequent calculi to be studied in this paper, we will not consider rules for type-formers such as Π-types, Σ-types and identity types, but rather concentrate on the core structural rules of weakening, projection and substitution. The combinatorics arising from just these rules is particularly elegant; we will see that in a freely-generated theory of this kind, the shape of derivable judgements is controlled by suitably decorated *heaps*. A heap is a finite tree with a total order on its nodes refining the tree order. This structure is common throughout logic and computer science, and the manner in which it appears here is highly reminiscent of its role in the study of (logical) game semantics and innocent strategies [9]. We hope to explore this link further in future work.

Beyond elucidating a structure which we believe to be interesting in its own right, the approach taken in this paper will also enable the analysis of dependent type theory using the tools of *combinatorial category theory*. This is a particular strand of category theory, growing out of Joyal's work [15], which has found recent applications [16,18,25,26] in taming some of the complexities of higher-dimensional category theory. A central theme in combinatorial category theory is the study of monads possessing abstract categorical properties that allow them to be seen as fundamentally combinatorial in nature. It turns out that the monad for dependent type theories is of this kind. More precisely, it is *local right adjoint* or *familially representable* in the sense of [4,17,21]: and this permits the application of a rich body of theory [2,18,24,25] concerning such monads to the study of dependent type theories. It is beyond the scope of this paper to investigate these connections in detail but let us mention two applications we intend to pursue in future work; see Section 8 below for a more detailed sketch of these applications.

Firstly, we will apply the results of [25] to describe a *nerve functor* for dependent sequent calculi: thus, a fully faithful embedding of the category of dependent sequent calculi into a presheaf category, together with a characterisation of the objects in the image. Nerve functors commonly serve to embed algebraic objects into geometric settings, so that in this case we may reasonably hope to find an implicit geometry of dependent sequent calculi.

Secondly, we will provide a new take on the *coherence problem* [11] for dependent type theory. We will do so by lifting the monad for dependent type theories to a 2-monad on a presheaf 2-category, and