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Polyhedral divisors and torus actions of complexity one over arbitrary fields

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ABSTRACT

We show that the presentation of affine $\mathbb{T}$-varieties of complexity one in terms of polyhedral divisors holds over an arbitrary field. We also describe a class of multigraded algebras over Dedekind domains. We study how the algebra associated with a polyhedral divisor changes when we extend the scalars. As another application, we provide a combinatorial description of affine $\mathbf{G}$-varieties of complexity one over a field, where $\mathbf{G}$ is a (not necessarily split) torus, by using elementary facts on Galois descent. This class of affine $\mathbf{G}$-varieties is described via a new combinatorial object, which we call (Galois) invariant polyhedral divisor.

0. Introduction

In this paper, we are interested in a combinatorial description of multigraded normal affine algebras of complexity one. From a geometrical viewpoint, these algebras are related to the classification of algebraic torus actions of complexity one on affine varieties. Let k be a field. Consider a split algebraic torus $\mathbb{T}$ over k. Recall that a $\mathbb{T}$-variety is a normal variety over k endowed with an effective $\mathbb{T}$-action. There exist several combinatorial descriptions of $\mathbb{T}$-varieties in term of the convex geometry. See [8,25,9,12] for the Dolgachev–Pinkham–Demazure (D.P.D.) presentation, [20,30,31] for toric case and complexity one case, and [1–3] for higher complexity. Most classical works on $\mathbb{T}$-varieties require the ground field k to be algebraically closed of characteristic zero. It is worthwhile mentioning that the description of affine $\mathbb{G}_m$-varieties [9] due to Demazure holds over any field.

Let us list the most important results of the paper.

- The Altmann–Hausen presentation of affine $\mathbb{T}$-varieties of complexity one in terms of polyhedral divisor holds over an arbitrary field, see Theorem 4.3.

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– This description holds as well for an important class of multigraded algebras over Dedekind domains, see Theorem 2.5.
– We study how the algebra associated with a polyhedral divisor changes when we extend the scalars, see Proposition 2.12 and Theorem 3.9.
– As another application, we provide a combinatorial description of affine **G**-varieties of complexity one, where **G** is a (not necessarily split) torus over k, by using elementary facts on Galois descent. This class of affine **G**-varieties is classified via a new combinatorial object, which we call a (Galois) invariant polyhedral divisor, see Theorem 5.10.

Let us discuss these results in more detail. We start with a simple case of varieties with an action of a split torus. Recall that a split algebraic torus $\mathbb{T}$ of dimension n defined over k is an algebraic group over k isomorphic to $\mathbb{G}_m^n$, where $\mathbb{G}_m = \mathbb{G}_{m,k}$ is the multiplicative algebraic group $\operatorname{Spec} k[t, t^{-1}]$. Let $M = \operatorname{Hom}(\mathbb{T}, \mathbb{G}_m)$ be the character lattice of the torus $\mathbb{T}$. Then defining a $\mathbb{T}$-action on an affine variety X is equivalent to fixing an M-grading on the algebra $A = k[X]$, where $k[X]$ is the coordinate ring of X. Following the classification of affine $\mathbb{G}_m$-surfaces [11], we say as in [22, 1.1], that the M-graded algebra A is *elliptic* if the graded piece A_0 is reduced to k.

Multigraded affine algebras are classified via a numerical invariant called complexity. This invariant was introduced in [24] for the classification of homogeneous spaces under the action of a connected reductive group. Consider the field $k(X)$ of rational functions on X and its subfield K_0 of $\mathbb{T}$-invariant functions. The complexity of the $\mathbb{T}$-action on X is the transcendence degree of K_0 over the field k. Note that for the situation where k is algebraically closed, the complexity is also the codimension of the general $\mathbb{T}$-orbit in X (see [26]).

In order to describe affine $\mathbb{T}$-varieties of complexity one, we have to consider combinatorial objects coming from convex geometry and from the geometry of algebraic curves. Let C be a regular curve over k. A point of C is assumed to be a closed point, and in particular, not necessarily rational. Thus, the residue field extension of k at any point of C has finite degree.

To reformulate our first result, we recall some notation introduced in [1, Section 1]. Denote by $N = \operatorname{Hom}(\mathbb{G}_m, \mathbb{T})$ the lattice of one-parameter subgroups of the torus $\mathbb{T}$ which is the dual of the lattice M. Let $M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$, $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$ be the associated dual $\mathbb{Q}$-vector spaces of M, N, respectively, and let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. We can define as in [1] a Weil divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ with σ-polyhedral coefficients in $N_{\mathbb{Q}}$, called polyhedral divisor of Altmann–Hausen. More precisely, each $\Delta_z \subset N_{\mathbb{Q}}$ is a polyhedron with a tail cone σ (see 2.1) and $\Delta_z = \sigma$ for all but finitely many points $z \in C$. Denoting by κ_z the residue field of the point $z \in C$ and by $[\kappa_z : k] \cdot \Delta_z$ the image of Δ_z under the homothety of ratio $[\kappa_z : k]$, the sum

$$\deg \mathfrak{D} = \sum_{z \in C} [\kappa_z : k] \cdot \Delta_z$$

is a polyhedron in $N_{\mathbb{Q}}$. This sum may be seen as the finite Minkowski sum of all polyhedra $[\kappa_z : k] \cdot \Delta_z$ different from σ. Considering the dual cone $\sigma^\vee \subset M_{\mathbb{Q}}$ of σ, we define an evaluation function

$$\sigma^\vee \to \operatorname{Div}_{\mathbb{Q}}(C), \qquad m \mapsto \mathfrak{D}(m) = \sum_{z \in C} \min_{l \in \Delta_z} \langle m, l \rangle \cdot z$$

with values in the vector space $\operatorname{Div}_{\mathbb{Q}}(C)$ of Weil $\mathbb{Q}$-divisors over C. As in the classical case [1, 2.12] we introduce the technical condition of properness for the polyhedral divisor $\mathfrak{D}$ (see Definitions 2.2, 3.4, 4.2) that we recall thereafter.