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Two generalizations of projective modules and their applications

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ABSTRACT

Let R be a commutative ring, M be an R-module, and w be the so-called w-operation on R. Set $\mathfrak{S}_w = \{f \in R[X] \mid c(f)_w = R\}$, where $c(f)$ denotes the content of f. Let $R\{X\} = R[X]_{\mathfrak{S}_w}$ and $M\{X\} = M[X]_{\mathfrak{S}_w}$ be the w-Nagata ring of R and the w-Nagata module of M respectively. Then we introduce and study two concepts of w-projective modules and w-invertible modules, which both generalize projective modules. To do so, we use two main methods of which one is to localize at maximal w-ideals of R and the other is to utilize w-Nagata modules over w-Nagata rings. In particular, it is shown that an R-module M is w-projective of finite type if and only if $M\{X\}$ is finitely generated projective over $R\{X\}$; M is w-invertible if and only if $M\{X\}$ is invertible over $R\{X\}$. As applications, it is shown that R is semisimple if and only if every R-module is w-projective and that, in a Q_0-PVMR, every finite type semi-regular module is w-projective.

0. Introduction and preliminaries

Since the concept of semi-divisorial modules, which generalizes both divisorial modules and injective modules, was introduced by Glaz and Vasconcelos [6] and was modified to allow the semi-divisorial closure (or w-closure) by the first author, the so-called w-operation has proved to be useful in the study of multiplicative ideal theory and module theory. The most important classes of modules, from a homological point of view, are the projective, injective, and flat modules. The introduction of the w-operation in the class of injective (resp., flat) modules has been successful (for example, [4,14,29] (resp., [2])). Unlike these two classes, the introduction and study of the w-operation in the class of projective modules has been less discussed. In this paper, we introduce two generalizations, with respect to the w-operation, of projective modules and provide the first systematic treatment to study them.

Throughout, R denotes a commutative ring with identity 1 and $E(M)$ denotes the injective hull (or envelope) of an R-module M.

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Let J be an ideal of R. Following [31], J is called a *Glaz–Vasconcelos ideal* (a *GV-ideal* for short) if J is finitely generated and the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism. Note that the set $\mathrm{GV}(R)$ of GV-ideals of R is a multiplicative system of ideals of R. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

Thus $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. Now M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). A GV-torsion-free module M is called a *w-module* if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$. Then projective modules and reflexive modules are w-modules. In the recent paper [32], it was shown that flat modules are w-modules. For any GV-torsion-free module M,

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}$$

is a w-submodule of $E(M)$ containing M and is called the *w-envelope* of M. It is clear that a GV-torsion-free module M is a w-module if and only if $M_w = M$.

For easy reference, we list some of the results on GV-torsion (resp., GV-torsion-free) modules and w-modules which will be used frequently.

Lemma 0.1. *Let R be a commutative ring with identity. Then the following statements are satisfied:*

(1) ([29, Theorem 1.3]) *An R-module M is a* GV-*torsion module if and only if* $\mathrm{Hom}_R(M, N) = 0$ *for any* GV-*torsion-free module N; an R-module N is a* GV-*torsion-free module if and only if* $\mathrm{Hom}_R(M, N) = 0$ *for any* GV-*torsion module M.*
(2) ([29, Theorem 2.7]) *An R-module M is* GV-*torsion if and only if $M_{\mathfrak{m}} = 0$ for all maximal w-ideal $\mathfrak{m}$ of R.*
(3) ([29, Theorem 2.8]) *A* GV-*torsion-free R-module M is a w-module if and only if* $\mathrm{Ext}^1_R(A_w/A, M) = 0$ *for any* GV-*torsion-free R-module A, if and only if* $\mathrm{Ext}^1_R(C, M) = 0$ *for any* GV-*torsion R-module C.*

Note that in the language of torsion theories, the w-envelope for modules coincides with the $\mathrm{tor}_{\mathrm{GV}}$-injective envelope with respect to the torsion theory whose torsion modules are the GV-torsion modules and the torsion-free modules are the GV-torsion-free modules. Thus the w-operation theory is a bridge closely connecting torsion theory with multiplicative ideal theory.

Many methods on multiplicative ideal theory are successful in characterizing and constructing some integral domains. It is natural to hope that these methods have good behavior for commutative rings with zero divisors. Comparing with the notion of w-modules that appeared in [28], when we use new definition of w-modules, we see that much discussion on domains has been generalized to commutative rings with zero divisors, for example, see [29,31]. We also notice, with the help of the w-operation and w-modules, that the notion of w-projective modules that is introduced in [24] is a generalization of t-invertibility and projectivity over domains. In [24] a module M is said to be *w-projective* if M is torsion-free of finite type and $M_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}$ for every maximal w-ideal $\mathfrak{m}$ of R. As characterizing a PVMD (a Prüfer v-multiplication domain) with t-invertibility, it was shown in [25] that a domain R is a PVMD if and only if every finite type torsion-free R-module is w-projective. Following Lucas [18], we study multiplicative ideal theory on a commutative ring with zero divisors by replacing the quotient field of a domain with the ring $Q_0(R)$ of finite fractions of an arbitrary commutative ring R. And it is shown that a commutative ring R is a Q_0-PVMR if and only if every finite type semi-regular submodule is w-projective.

Any undefined terminology is standard as in [5,22,23,30] or will be explained in the course of this paper.