Journal of Pure and Applied Algebra 219 (2015) 2139–2143

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

A bimodule approach to the strong no loop conjecture

Yang Han

KLMM, ISS, AMSS, Chinese Academy of Sciences, Beijing 100190, PR China

ARTICLE INFO

Article history:
Received 21 March 2014
Received in revised form 30 June 2014
Available online 4 August 2014
Communicated by S. Koenig

MSC:
16G10; 16E10

ABSTRACT

Let A be a finite-dimensional elementary k-algebra, where k is a field. For B a finite-dimensional k-algebra, the Hattori–Stallings trace is studied at the level of projective B-modules that are A–B-bimodules. A bimodule characterization of the projective dimension of a simple A-module is provided. These results are applied to give an alternative proof of Igusa–Liu–Paquette Theorem, i.e., the strong no loop conjecture for finite-dimensional elementary k-algebras.

1. Introduction

Throughout this paper, k is a field and all algebras are associative k-algebras with identity, unless otherwise indicated. Global dimension is a quite important homological invariant of an algebra or a ring. For example, the famous Hilbert syzygy theorem says that the global dimension of the polynomial algebra in n variables is n. Sometimes we do not pay much attention to the precise value of the global dimension but in its finiteness. In fact, the finiteness of global dimension plays an important role in representation theory of algebras. For instance, the bounded derived category of a finite-dimensional algebra has Auslander–Reiten triangles if and only if the algebra is of finite global dimension [10,11]. Moreover, there are some well-known conjectures related to the finiteness of global dimension such as the *no loop conjecture* which will be stated in the sequel, the *Cartan determinant conjecture* — the determinant of the Cartan matrix of an artin ring of finite global dimension is 1 (Ref. [5]), the *Hochschild homology dimension conjecture* — a finite-dimensional algebra is of finite global dimension if its Hochschild homology dimension is finite [8]. Associated with a finite-dimensional elementary algebra is its quiver [1, Theorem III.1.9]. The finiteness of the global dimension of a finite-dimensional elementary algebra, or equivalently a bound quiver algebra, is closely related to the combinatorics of its quiver. A bound quiver algebra without oriented cycles in its quiver must be of finite global dimension [4]. Obviously, the converse is not true in general. Nevertheless, a bound quiver algebra of finite global dimension must have no loops and 2-truncated cycles [2]. The former is due to the following no loop conjecture:

E-mail address: hany@iss.ac.cn.

http://dx.doi.org/10.1016/j.jpaa.2014.07.028
0022-4049/

No loop conjecture. *Let A be an artin algebra of finite global dimension. Then* $\mathrm{Ext}^1_A(S,S)=0$ *for all simple A-modules S.*

The no loop conjecture was first explicitly established for artin algebras of global dimension two [6, p. 378, Proposition]. For finite-dimensional elementary algebras, as shown in [13], this can be easily derived from an earlier result of Lenzing [16]. A stronger version of the no loop conjecture is the following strong no loop conjecture:

Strong no loop conjecture. *Let A be an artin algebra and S a simple A-module of finite projective dimension. Then* $\mathrm{Ext}^1_A(S,S)=0$.

The strong no loop conjecture is due to Zacharia [13], which is also listed as a conjecture in Auslander–Reiten–Smalø's book [1, p. 410, Conjecture (7)]. For finite-dimensional elementary algebras, particularly, for finite-dimensional algebras over an algebraically closed field, it was proved by Igusa, Liu and Paquette in [14]. Before it, some special cases had been solved in [3,7,15,17–19,21].

In this paper, we shall study the Hattori–Stallings traces of projective bimodules and one-sided projective bimodules as one-sided projective modules (Proposition 2 and Proposition 3) and provide a bimodule characterization of the projective dimension of a simple module (Proposition 4). Furthermore, we shall apply these results to give an alternative proof of Igusa–Liu–Paquette Theorem (Theorem 1), i.e., the strong no loop conjecture for finite-dimensional elementary algebras.

2. Hattori–Stallings traces

Let A be a ring with identity. Denote by $\mathrm{mod}\,A$ the category of finitely generated right A-modules, and by $\mathrm{proj}\,A$ the full subcategory of $\mathrm{mod}\,A$ consisting of all finitely generated projective right A-modules.

For each $P\in\mathrm{proj}\,A$, there is an isomorphism of abelian groups

$$\phi_P : P\otimes_A \mathrm{Hom}_A(P,A)\to \mathrm{End}_A(P)$$

defined by $\phi_P(p\otimes f)(p')=pf(p')$ for all $p,p'\in P$ and $f\in\mathrm{Hom}_A(P,A)$. There is also a homomorphism of abelian groups

$$\psi_P : P\otimes_A \mathrm{Hom}_A(P,A)\to A/[A,A]$$

defined by $\psi_P(p\otimes f)=\overline{f(p)}$ for all $p\in P$ and $f\in\mathrm{Hom}_A(P,A)$. Here, $[A,A]$ is the additive subgroup of A generated by all commutators $[a,b]:=ab-ba$ with $a,b\in A$, and $\bar{a}\in A/[A,A]$ denotes the equivalence class of a in A. It is well-known that the abelian group $A/[A,A]$ is isomorphic to the zero-th Hochschild homology group $HH_0(A)$ of A. The homomorphism of abelian groups

$$\mathrm{tr}_P:=\psi_P\phi_P^{-1} : \mathrm{End}_A(P)\to A/[A,A]$$

is called *the Hattori–Stallings trace of P*.

Hattori–Stallings trace has the following properties:

Proposition 1. *(See Hattori [12], Stallings [20], Lenzing [16].) Let* $P,P',P''\in\mathrm{proj}\,A$.

(HS1) *If* $f\in\mathrm{End}_A(P)$ *and* $g\in\mathrm{Hom}_A(P,P')$ *is an isomorphism then* $\mathrm{tr}_P(f)=\mathrm{tr}_{P'}(gfg^{-1})$.
(HS2) *If* $f,f'\in\mathrm{End}_A(P)$ *then* $\mathrm{tr}_P(f+f')=\mathrm{tr}_P(f)+\mathrm{tr}_P(f')$.