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Exponentiation of commuting nilpotent varieties

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ABSTRACT

Let H be a linear algebraic group over an algebraically closed field of characteristic $p > 0$. We prove that any "exponential map" for H induces a bijection between the variety of r-tuples of commuting $[p]$-nilpotent elements in $Lie(H)$ and the variety of height r infinitesimal one-parameter subgroups of H. In particular, we show that for a connected reductive group G in pretty good characteristic, there is a canonical exponential map for G and hence a canonical bijection between the aforementioned varieties, answering in this case questions raised both implicitly and explicitly by Suslin, Friedlander, and Bendel.

Let H be a linear algebraic group over an algebraically closed field k of characteristic $p > 0$. Let $H_{(r)}$ denote the r-th Frobenius kernel of H, and $\mathfrak{h}$ its Lie algebra. There is a $[p]$-mapping on $\mathfrak{h}$ which sends $x \mapsto x^{[p]}$, and we set $\mathcal{N}_1(\mathfrak{h})$ to be the restricted nullcone of $\mathfrak{h}$, which consists of those x for which $x^{[p]} = 0$. Let $\mathcal{C}_r(\mathcal{N}_1(\mathfrak{h}))$ denote the variety of r-tuples of commuting elements in $\mathcal{N}_1(\mathfrak{h})$, while $\mathrm{Hom}_{gs/k}(\mathbb{G}_{a(r)}, H)$ is the affine variety of height r infinitesimal one-parameter subgroups of H (it is the set of k-points of the affine scheme in [16, Theorem 1.5]). It was shown by Suslin, Friedlander, and Bendel in [16] and [17] that this last variety is homeomorphic to the cohomological variety of $H_{(r)}$, thereby establishing its importance within representation theory, specifically within the study of support varieties for modules.

Our aim in this paper is to provide a better understanding of the relationship between $\mathcal{C}_r(\mathcal{N}_1(\mathfrak{h}))$ and $\mathrm{Hom}_{gs/k}(\mathbb{G}_{a(r)}, H)$. We give a definition of what we call an *exponential map* for H, and show that any such map, if it exists, induces a bijection between these varieties. In particular, we show that if G is a connected reductive group over k in *pretty good* characteristic (see Section 1), then there is a canonical exponential map for G, and hence a canonical bijection between $\mathcal{C}_r(\mathcal{N}_1(\mathfrak{g}))$ and $\mathrm{Hom}_{gs/k}(\mathbb{G}_{a(r)}, G)$, extending results along these lines found in [16,9,14]. We also give an example of a linear algebraic group H and an $r > 1$ for which $\mathcal{C}_r(\mathcal{N}_1(\mathfrak{h}))$ and $\mathrm{Hom}_{grp/k}(\mathbb{G}_{a(r)}, H)$ are varieties which have different dimensions.

The importance of obtaining an explicit description of $\mathrm{Hom}_{grp/k}(\mathbb{G}_{a(r)}, H)$ is (from our point of view) primarily because it is a necessary step in extending support variety computations such as those in [10,5,13].

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However, the results in this paper also have a new application to recent work by Friedlander [6], in which the author defines support varieties for rational H-modules (that is, support varieties directly defined for H rather than for the Frobenius kernels of H).

This paper is organized as follows. After recalling relevant concepts in Section 1, we give in Section 2 our definition of an exponential map on $\mathcal{N}_1(\mathfrak{h})$ and prove that it induces the variety bijection mentioned above. Section 3 contains the strongest results which are available only in the case of connected reductive groups. Our final section uses the work in Section 3 to give a simplified proof that one can achieve *saturation* for a connected reductive group in good characteristic. This notion is due to Serre, and involves assigning to each p-unipotent element $g \in G$ a one-parameter subgroup of G whose image contains g, and which can be specified in some canonical way. Seitz explored saturation in [11] and proved that it can be achieved for any such G.

1. Preliminaries and notation

We fix k to be an algebraically closed field of characteristic $p > 0$. If H is an affine algebraic group over k, then it is also an affine group scheme over k, and by abuse of notation we will use H to denote both the scheme and the group of k-points of the scheme (as will be clear by the context). Let $\mathcal{U}_1(H)$ denote the closed subset of p-unipotent elements in H (i.e. the unipotent variety of H). For $g, h \in H$, $X \in \mathfrak{h}$, we write $g \cdot h = ghg^{-1}$, while $g \cdot X$ denotes the adjoint action. The centralizer of g is $C_H(g)$, $C_H(X)$ is the stabilizer of X in H, $C_{\mathfrak{h}}(X)$ is the centralizer of X in $\mathfrak{h}$, and $C_{\mathfrak{h}}(g)$ are those elements in $\mathfrak{h}$ which are stabilized by g.

For any affine group scheme H (i.e. not necessarily coming from an algebraic group), we write $k[H]$ for its coordinate ring. We denote by $Dist(H)$ its algebra of distributions (see [8, I.7]), and by $\mathfrak{h}$ its Lie algebra. Let $H_{(r)}$ denote the r-th Frobenius kernel of H (see [8, I.9] regarding how to define such a kernel without specifying an $\mathbb{F}_p$-structure on H). We then have that $Dist(H_{(r)}) \subseteq Dist(H_{(r+1)})$, and $Dist(H) = \bigcup_{r \geq 1} Dist(H_{(r)})$. We also recall that $Dist(H_{(1)})$ is isomorphic as a Hopf algebra to the restricted enveloping algebra of $\mathfrak{h}$ [8, I.9.6(4)].

If φ is a homomorphism of affine group schemes from H_1 to H_2, then it induces a homomorphism of Hopf algebras from $Dist(H_1)$ to $Dist(H_2)$, and we will denote this map by $d\varphi$. By abuse of notation we will also use $d\varphi$ in the more standard way to denote the differential of φ, that is the induced map from $\mathfrak{h}_1$ to $\mathfrak{h}_2$. In fact, this map on Lie algebras can really be viewed as a restriction of the map on distribution algebras by the comments above.

The additive group $\mathbb{G}_a$ has coordinate algebra $k[\mathbb{G}_a] \cong k[t]$, and $Dist(\mathbb{G}_a)$ is spanned by the elements $\frac{d}{dt}^{(j)}$, where

$$\frac{d}{dt}^{(j)}(t^i) = \delta_{ij}.$$

This notation is not standard. For example, Jantzen denotes the element $\frac{d}{dt}^{(j)}$ as γ_j in [8, I.7.8]. If we set $u_j = \frac{d}{dt}^{(p^j)}$, and if m is an integer with p-adic expansion

$$m = m_0 + m_1 p + \cdots + m_q p^q, \quad 0 \leq m_i < p,$$

then

$$\frac{d}{dt}^{(m)} = \frac{u_0^{m_0} \cdots u_q^{m_q}}{m_0! \cdots m_q!}$$

Therefore $Dist(\mathbb{G}_a)$ is generated as an algebra over k by the set $\{u_j\}_{j \geq 0}$, while $Dist(\mathbb{G}_{a(r)})$ is generated by the subset where $j < r$. Also, for any affine group scheme H, a homomorphism from $\mathbb{G}_{a(r)}$ to H is