Journal of Pure and Applied Algebra 219 (2015) 2245–2259

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

An algebraic study of linkages with helical joints

Hamid Ahmadinezhad[a], Zijia Li[a,*], Josef Schicho[b]

[a] Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenberger Str. 69, A-4040 Linz, Austria
[b] Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Altenberger Str. 69, A-4040 Linz, Austria

ARTICLE INFO

Article history:
Received 4 December 2013
Received in revised form 26 June 2014
Available online 6 August 2014
Communicated by R. Vakil

MSC:
70B15; 14Q99; 14H99

ABSTRACT

Methods from algebra and algebraic geometry have been used in various ways to study linkages in kinematics. These methods have failed so far for the study of linkages with helical joints (joints with screw motion), because of the presence of some non-algebraic relations. In this article, we explore a delicate reduction of some analytic equations in kinematics to algebraic questions via a theorem of Ax. As an application, we give a classification of mobile closed 5-linkages with revolute, prismatic, and helical joints.

1. Introduction

Linkages, and in particular closed linkages, are a crucial object of study in the modern theory of kinematics. The use of algebra and geometry for studying linkages is very classical and goes back to Sylvester, Kempe, Cayley and Chebyshev.

A linkage, as appearing in robotics/mechanical engineering, biology, as well as modelling of molecules in chemistry, etc., is a mechanical structure that consists of a finite number of rigid bodies – its *links* – and a finite number of *joints* that connect the links together, so that they possibly produce a motion. A linkage is called *closed* if its number of links and joints are equal and they are connected cyclically. We consider four types of joints:

(R) revolute joints: allow rotations around a fixed axes;
(P) prismatic joints: allow translations in a fixed direction;
(C) cylindrical joints: allow rotations around a fixed axes and translations in the direction of the axes;

* Corresponding author.
E-mail addresses: hamid.ahmadinezhad@oeaw.ac.at (H. Ahmadinezhad), zijia.li@oeaw.ac.at (Z. Li), josef.schicho@risc.jku.at (J. Schicho).

http://dx.doi.org/10.1016/j.jpaa.2014.07.034
0022-4049/

(H) helical joints: allow the motions of a cylindrical joint where the rotation angle and the translation length are coupled by a linear equation.

We will use the notation R-joint for a revolute joint, and similarly for other types. Note that the dimension of the set of allowed motions (the degree of freedom) is 1 for joints of type R, P, and H, and it is 2 for C-joints.

The *configuration set* of a closed linkage L, denoted by K_L, is the set of possible simultaneous motions of all joints (see Definition 1 for a precise description). The dimension of K_L is called the *mobility* of L, and L is called *mobile* if the mobility is positive.

It is known that K_L can be described by analytic equations; see [14], page 356. If there are no H-joints, then we also have a description by algebraic equations. We refer to [13] for a historic overview of the use of geometric algebra in kinematics. This is the subject that has attracted algebraists the most. It should be mentioned, however, that also number theory has been used for studying linkages: in [9], reduction modulo prime numbers are considered in order to construct a new family of Stewart–Gough platforms. Below, in Section 2 we briefly explain the algebraic setup, as well as the theory of bonds, a rather new combinatorial technique that has shown to be very useful for analysing closed linkages with R-joints [10]. We also explain the analytical relations in the presence of H-joints. There are also other (numerical) algebraic methods that are applied in kinematics, see for example [3,7,15].

A closed linkage with n joints, where all joints are R-joints, is denoted by nR-linkage. We denote by n-linkage a linkage with n joints where no information on the type of joints is specified. It is easy to imagine that a 3R-linkage does not have a motion, and hence its configuration set is trivial. On the other hand a generic nR-linkage for $n \geq 7$ has positive mobility (see [14], page 356), and hence there is not much to study. So the interesting cases are when $n = 4, 5$ or 6. Nowadays we have a full classification of 4R- and 5R-linkages with mobility one ($\dim K_L = 1$), and we know many cases for $n = 6$ (see [8]). It is an open research problem to classify all 6R-linkages. These classification problems are considered by algebraists. Because of the nature of other types of linkages it seemed difficult, or rather impossible, to be able to use any of the present algebraic techniques for linkages with H-joints.

1.1. What is new in this article?

As a main result, we show that unexpected mobility of a linkage with H-joints, i.e., a mobility that is strictly bigger than predicted by the Grübler–Kutzbach–Chebyshev formula which simply counts parameters and equational restrictions, can always be explained algebraically. Let L be a linkage with H-joints, and let L' be the linkage obtained from L by replacing all H-joints by C-joints. It is clear that the configuration set of L is a subset of the configuration set of L'. The relation between the configuration set of L' and the Zariski closure of the configuration set of L will be made very precise (Theorem 9), with the help of Ax's theorem [1] on the transcendence degree of function fields with exponentials. Note that Ax's theorem is originally about Schanuel's conjecture in number theory and has no apparent connections to kinematics.

The mobile linkages with 4 joints of type H, R, or P have been classified in [6]. Here (more precisely in Theorem 10) we give a classification of mobile linkages with 5 joints of type H, R, or P. Using our main result, we reduce to linkages with joints of type R or P only. The classification of mobile 5R-linkages has been done in [12], but for linkages with both R-joints and P-joints, we could not find a complete classification in the literature. On the other hand, this classification is not difficult when we use the theory of bonds, so we also give it in here (Theorem 6).

1.2. Structure of the paper

In Section 2 we set up a mathematical language to describe and to analyse linkages with arbitrary types of joints, we recall the theory of bonds for R-joints and we introduce the adaptations of this theory that