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Moduli spaces of linear bundles on hyperquadrics

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A B S T R A C T

In this paper, we will deal with semistable rank $2l$ vector bundles on odd dimensional hyperquadrics $Q_{2l+1} \subset \mathbb{P}^{2l+2}$ given as the cohomology bundles of linear monads. Firstly we prove that symplectic special linear bundles are stable. Then, inside the Maruyama scheme we consider the locus $ML_{Q_{2l+1}}(k)$ parameterizing rank $2l$ linear bundles on Q_{2l+1} with second Chern class k and we analyze its geometric properties. We prove that the moduli space $ML_{Q_{2l+1}}(1)$ is smooth, irreducible of dimension $2l^2 + 5l + 2$ and that for any $l \geq 2$ and $k \geq 2$, the moduli space $ML_{Q_{2l+1}}(k)$ is singular.

1. Introduction

Let V be a $\mathbb{C}$-vector space of dimension $2l+3$, $l \geq 1$ and let $Q_{2l+1} \subset \mathbb{P}^{2l+2} = \mathbb{P}(V)$ be a smooth quadric hypersurface. The goal of this paper is to study the moduli space of stable rank $2l$ vector bundles on Q_{2l+1} defined as the cohomology bundle of a monad of the following type

$$0 \longrightarrow U_1 \otimes \mathcal{O}_{Q_{2l+1}}(-1) \xrightarrow{A} U_2 \otimes \mathcal{O}_{Q_{2l+1}} \xrightarrow{B^t} U_3 \otimes \mathcal{O}_{Q_{2l+1}}(1) \longrightarrow 0$$

where $A \in U_1^* \otimes U_2 \otimes V^*$ and $B^t \in U_2^* \otimes U_3 \otimes V^*$, U_1, U_2 and U_3 being $\mathbb{C}$-vector spaces of dimensions k, $2k+2l$ and k respectively; that is A and B are $(2l+2k) \times k$ matrices with linear entries. Monads were first introduced by Horrocks who showed that all vector bundles E on $\mathbb{P}^3$ can be obtained as the cohomology bundle of a monad of the following kind:

$$0 \longrightarrow \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(a_i) \longrightarrow \bigoplus_j \mathcal{O}_{\mathbb{P}^3}(b_j) \longrightarrow \bigoplus_n \mathcal{O}_{\mathbb{P}^3}(c_n) \longrightarrow 0.$$

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Since then, monads appear in a wide variety of contexts within algebraic-geometry, like the construction of locally free sheaves on a projective variety X, the classification of space curves in $\mathbb{P}^3$ and surfaces in $\mathbb{P}^4$, etc. In this paper, we will deal with the simplest monads, namely monads defined by matrices with linear entries, to construct indecomposable vector bundles on odd dimensional hyperquadrics $Q_{2l+1} \subset \mathbb{P}^{2l+2}$. Since the middle of the last century, the existence of indecomposable vector bundles E on a projective variety X has been a challenging problem which has attracted the attention of many geometers. This existence becomes more difficult when the rank r of E is small compared to the dimension of X. In general, there are few examples of indecomposable rank r, $1 < r < n$, vector bundles E on n-dimensional projective varieties X and, in addition Hartshorne conjectured that there are no rank two vector bundles on $\mathbb{P}^m$ for $m \geq 7$ [5]. In this paper, we focus our attention on rank $2l$ vector bundles on hyperquadrics $Q_{2l+1} \subset \mathbb{P}^{2l+2}$ which will be obtained as a cohomology bundles of monads with linear entries and for that reason we call them linear bundles. Once the existence of rank $2l$ linear bundles on Q_{2l+1} is stated, we would like to study them from the moduli point of view. This compels us to study the (semi-)stability of the linear bundles E on Q_{2l+1}, $l \geq 1$, a question that in general remains open (see Section 6). Indeed, in general it is a difficult question to determine whether a vector bundle is stable or not due to the lack of useful stability criterion and this difficulty increases a lot while dealing with higher rank vector bundles. In this paper, we will prove that any linear bundle on Q_3 and on Q_5 is stable (Proposition 3.3) and that rank $2l$ symplectic special linear bundles on Q_{2l+1} (Definition 2.5) are stable as well (Theorem 3.4). Since all linear bundles will turn to be semistable, inside the Maruyama scheme we can consider the locus $ML_{Q_{2l+1}}(k)$ parameterizing linear bundles on Q_{2l+1} and we ask for its geometric properties: irreducibility, smoothness, dimension, rationality, etc. In this paper we will describe the tangent space at a point parameterizing a symplectic special linear bundle (Theorem 4.5) and we will analyze the smoothness of the moduli space $ML_{Q_{2l+1}}(k)$. It turns out that the moduli space $ML_{Q_{2l+1}}(1)$ is smooth, irreducible of dimension $2l^2+5l+2$ (Proposition 5.4) and that for any $l \geq 2$ and $k \geq 2$, the moduli space $ML_{Q_{2l+1}}(k)$ is singular (Theorem 5.5). We will end the paper with some open problems and conjectures concerning the moduli spaces $ML_{Q_{2l+1}}(k)$.

Next we outline the structure of the paper. In Section 2, we fix notation and we briefly recall the definition and basic properties of linear bundles on hyperquadrics needed later on. The main goal of Section 3 is to prove the stability (in the sense of Mumford–Takemoto) of special linear bundles on Q_{2l+1} with a symplectic structure (Theorem 3.4) and of any linear bundle on Q_3 and Q_5 (Proposition 3.3).

Let us denote by $ML_{Q_{2l+1}}(k)$ the open subset of the Maruyama scheme of semistable coherent sheaves on Q_{2l+1} with Chern polynomial $c_t(E) = \frac{1}{(1-e_1t)^k(1+e_1t)^k}$ being $e_1 \in H^2(Q_{2l+1};\mathbb{Z})$ a generating class. In Section 4, we compute the dimension of the Zariski tangent space of $ML_{Q_{2l+1}}(k)$ at points parameterizing symplectic special linear bundles (Theorem 4.5). The smoothness of the moduli space $ML_{Q_{2l+1}}(k)$ is discussed in Section 5. For $k = 1$, the moduli space is smooth while for $l > 1$ and $k > 1$, the moduli space turns out to be singular. In fact, the singular locus of $ML_{Q_{2l+1}}(k)$, $l, k \geq 2$, contains at least the points parameterizing symplectic special linear bundles (Theorem 5.5). In Section 6 we end with some problems/conjectures which naturally arise in our context.

Notation. Throughout this paper we will work over the complex numbers $\mathbb{C}$. We will not distinguish between a vector bundle and its locally free sheaf of sections and we will use the definition of stability and semistability due to Mumford–Takemoto [8]. For any coherent sheaf F on Q_n we use the abbreviations $F(d) := F \otimes \mathcal{O}_{Q_n}(d)$, $H^i(F) := H^i(Q_n, F)$ and $h^iF = \dim_{\mathbb{C}} H^i(Q_n, F)$.

2. Preliminaries

In this section, we give the definition and basic properties of linear bundles on odd dimensional hyperquadrics, and we introduce a particular case of linear bundles that we will call special linear bundles.