Journal of Pure and Applied Algebra 219 (2015) 429–448

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Vanishing of Tor, and why we care about it

Olgur Celikbas [a], Roger Wiegand [b,*,1]

[a] *Department of Mathematics, University of Missouri, Columbia, MO 65211, USA*
[b] *Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588, USA*

ARTICLE INFO

Article history:
Available online 21 May 2014

This paper is dedicated to memory of Hans-Bjørn Foxby in recognition of his huge impact on homological algebra, module theory, and commutative algebra

MSC:
13D03

ABSTRACT

Given finitely generated modules M and N over a local ring R, the tensor product $M \otimes_R N$ typically has nonzero torsion. Indeed, the assumption that the tensor product is torsion-free influences the structure and vanishing of the modules $\operatorname{Tor}_i^R(M,N)$ for all $i \geq 1$. In turn, the vanishing of $\operatorname{Tor}_i^R(M,N)$ imposes restrictions on the depth properties of the modules M and N. These connections made their first appearance in Auslander's 1961 paper "Modules over unramified regular local rings". We will survey the literature on these topics, with emphasis on progress during the past twenty years.

1. Introduction

In the first paragraph of his famous paper *Modules over unramified regular local rings* [3], Auslander wrote: "The main object of study is what it means about two modules A and B over an unramified regular local ring to assert that the torsion submodule of $A \otimes B$ is zero." Indeed, Auslander showed, assuming that A and B are finitely generated nonzero modules over such a ring R, that both A and B must be torsion-free, that $\operatorname{Tor}_i^R(A,B) = 0$ for all $i \geq 1$, and that $\operatorname{pd}_R A + \operatorname{pd}_R B = \operatorname{pd}_R(A \otimes_R B) < \dim R$, unless R is a field. In this paper we will discuss some of what has been done on these topics, for more general local rings, in the 50+ years since Auslander's paper. Our main theme is the same as Auslander's: we assume that the tensor product of two nonzero modules is torsion-free (or is reflexive, or satisfies a certain Serre condition, ...), and then see what we can learn about the modules M and N. The vanishing of $\operatorname{Tor}_i^R(M,N)$, for all $i \geq 1$ (or, in some cases, for all $i \gg 0$) will play a pivotal role, as it did in Auslander's paper.

1.1. Notation and assumptions

The notation $(R, \mathfrak{m}, k)$ indicates that R is a local (commutative and Noetherian) ring with maximal ideal $\mathfrak{m}$ and residue field k. The *torsion* submodule of an R-module M is the kernel $\top M$ of the natural map

* Corresponding author.
E-mail addresses: celikbaso@missouri.edu (O. Celikbas), rwiegand1@math.unl.edu (R. Wiegand).

1 Wiegand's research was partially supported by a Simons Collaboration Grant 20913.

http://dx.doi.org/10.1016/j.jpaa.2014.05.003
0022-4049/

$M \to \mathrm{Q}(R) \otimes_R M$, where $\mathrm{Q}(R) = \{\text{non-zerodivisors}\}^{-1}R$, the total quotient ring of R. The module M is *torsion* provided $\top M = M$ and *torsion-free* provided $\top M = 0$. The torsion-free module $M/\top M$ is denoted $\perp M$.

2. Rigidity

We say that the pair (M, N) of finitely generated R-modules is *rigid* provided the vanishing of $\mathrm{Tor}_j^R(M, N)$ for some $j \geq 1$ forces $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq j$; and M is *rigid* provided the pair (M, N) is rigid for every finitely generated N. In the following theorem, we do not assume that the ring is a regular local ring, but the argument is essentially the same as Auslander's proof of [3, Lemma 3.1].

Theorem 2.1 *(Auslander, 1961). Let M and N be nonzero finitely generated modules over a reduced local ring R. Assume that either $\perp M$ or $\perp N$ is rigid and that $M \otimes_R N$ is torsion-free. Then:*

(i) Both M and N are torsion-free.
(ii) $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

We'll state the first step of the proof as a lemma, since it will come up several times in this survey.

Lemma 2.2. *Let M and N be finitely generated modules over a local ring R, and suppose that $M \otimes_R N$ is torsion-free. Then the natural maps in the diagram*

$$\begin{array}{ccc} M \otimes_R N & \longrightarrow & \perp M \otimes_R N \\ \downarrow & & \downarrow \\ M \otimes_R \perp N & \longrightarrow & \perp M \otimes_R \perp N \end{array}$$

are all isomorphisms. In particular, all four modules are torsion-free.

Proof. Tensoring the short exact sequence

$$0 \to \top M \to M \to \perp M \to 0 \tag{2.1}$$

with N, we obtain an exact sequence

$$\top M \otimes_R N \xrightarrow{\alpha} M \otimes_R N \xrightarrow{\beta} \perp M \otimes_R N \to 0.$$

Since $\top M \otimes_R N$ is torsion and $M \otimes_R N$ is torsion-free, α must be the zero-map, and hence β (the top arrow in the commutative diagram) is an isomorphism. By symmetry, so is the left-hand arrow. Now we know that $M \otimes_R \perp N$ is torsion-free, and, on tensoring (2.1) with $\perp N$, we see by the same argument that the bottom arrow is an isomorphism. The right-hand arrow is an isomorphism because the diagram commutes. □

Proof of Theorem 2.1. Using the fact that $\mathrm{Q}(R)$ is a direct product of fields, we embed $\perp M$ and $\perp N$ into finitely generated free modules F and G, respectively, and get short exact sequences:

$$0 \to \perp M \to F \to U \to 0 \tag{2.2}$$

$$0 \to \perp N \to F \to V \to 0 \tag{2.3}$$

Supposing that $\perp M$ is rigid, we apply $\perp M \otimes_R -$ to (2.3) to obtain an injection $\mathrm{Tor}_1^R(\perp M, V) \hookrightarrow \perp M \otimes_R \perp N$. Since $\mathrm{Tor}_1^R(\perp M, V)$ is torsion (again, because $\mathrm{Q}(R)$ is a direct product of fields) and $\perp M \otimes_R \perp N$