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The direct limit closure of perfect complexes [☆]

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In loving memory of Hans-Bjørn Foxby—our teacher, colleague, and friend

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ABSTRACT

Every projective module is flat. Conversely, every flat module is a direct limit of finitely generated free modules; this was proved independently by Govorov and Lazard in the 1960s. In this paper we prove an analogous result for complexes of modules, and as applications we reprove some results due to Enochs and García Rozas and to Neeman.

1. Introduction

Let R be a ring. In contrast to the projective objects in the category of R-modules, i.e. the projective R-modules, the projective objects in the category of R-complexes are not of much utility; indeed, they are nothing but contractible (split) complexes of projective R-modules. In the category of complexes, the relevant alternative to projectivity—from the homological point of view, at least—is semi-projectivity. A complex P is called *semi-projective* (or *DG-projective*) if the total Hom functor $\operatorname{Hom}(P,-)$ preserves surjective quasi-isomorphisms, i.e. surjective morphisms that induce isomorphisms in homology. The semi-projective complexes are exactly the cofibrant objects in the standard model structure on the category of complexes; see Hovey [11, §2.3]. Alternatively, a complex is semi-projective if and only if it consists of projective modules and it is K-projective in the sense of Spaltenstein [18]. The notion of semi-projectivity in the category of complexes extends the notion of projectivity in the category of modules in a natural and useful way: A module is projective if and only if it is semi-projective when viewed as a complex.

Similarly, a complex F is *semi-flat* if the total tensor product functor $-\otimes F$ preserves injective quasi-isomorphisms; equivalently, F is a complex of flat modules and K-flat in the sense of [18]. A module is flat if and only if it is semi-flat when viewed as a complex. Every semi-projective complex is semi-flat, and simple examples of semi-projective complexes are bounded complexes of finitely generated projective modules, also

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known as *perfect* complexes. The class of semi-flat complexes is closed under direct limits, and our main result, Theorem 1.1 below, shows that every semi-flat complex is a direct limit of perfect complexes. For modules, the theorem specializes to a classic result, proved independently by Govorov [9] and Lazard [13]: Every flat module is a direct limit of finitely generated free modules.

1.1. Theorem. *For an R-complex F the following conditions are equivalent.*

(*i*) *F is semi-flat.*
(*ii*) *Every morphism of R-complexes* $\varphi\colon N \to F$ *with N bounded and degreewise finitely presented admits a factorization,*

$$\begin{array}{ccc} N & \xrightarrow{\quad\varphi\quad} & F \\ & {\scriptstyle\kappa}\searrow \quad \nearrow{\scriptstyle\lambda} & \\ & L, & \end{array}$$

where L is a bounded complex of finitely generated free R-modules.
(*iii*) *There exists a set* $\{L^u\}_{u\in U}$ *of bounded complexes of finitely generated free R-modules and a pure epimorphism* $\coprod_{u\in U} L^u \to F$.
(*iv*) *F is isomorphic to a filtered colimit of bounded complexes of finitely generated free R-modules.*
(*v*) *F is isomorphic to a direct limit of bounded complexes of finitely generated free R-modules.*

The theorem is proved in Section 5. The terminology used in the statement is clarified in the sections leading up to the proof. In Section 4 we show that the finitely presented objects in the category of complexes are exactly the bounded complexes of finitely presented modules. Results of Breitsprecher [5] and Crawley-Boevey [6] show that the category of complexes is locally finitely presented, see Remark 4.7. Therefore, the equivalence of (*ii*), (*iii*), and (*iv*) follows from [6, (4.1)]. Furthermore, a result by Adámek and Rosický [1, Thm. 1.5] shows that (*iv*) and (*v*) are equivalent for quite general reasons; thus our task is to prove that the equivalent conditions (*ii*)–(*v*) are also equivalent to (*i*).

The characterization of semi-flat complexes in Theorem 1.1 opens to a study of the interplay between semi-flatness and purity in the category of complexes; this is the topic of Section 6. We show, for example, that a complex F is semi-flat if and only if every surjective quasi-isomorphism $M \to F$ is a pure epimorphism. This compares to Lazard's [13, Cor. 1.3] which states that a module F is flat if and only if every surjective homomorphism $M \to F$ is a pure epimorphism.

In the final Section 7, we use Theorem 1.1 to reprove a few results due to Enochs and García Rozas [7] and to Neeman [17]; our proofs are substantially different from the originals. In Theorem 7.3 we show that an acyclic semi-flat complex is a direct limit of contractible perfect complexes. Combined with a result of Benson and Goodearl [4] this enables us to show in Theorem 7.8 that a semi-flat complex of projective modules is semi-projective.

2. Complexes

In this paper R is a ring, and the default action on modules is on the left. Thus, R-modules are left R-modules, while right R-modules are considered to be (left) modules over the opposite ring R°. The definitions and results listed in this section are standard and more details can be found in textbooks, such as Weibel's [19], and in the paper [2] by Avramov and Foxby.

An R-complex M is a graded R-module $M = \coprod_{v\in\mathbb{Z}} M_v$ equipped with a differential, that is, an R-linear map $\partial^M\colon M \to M$ that satisfies $\partial^M\partial^M = 0$ and $\partial^M(M_v) \subseteq M_{v-1}$ for every $v \in \mathbb{Z}$. The homomorphism $M_v \to M_{v-1}$ induced by ∂^M is denoted ∂^M_v. Thus, an R-complex M can be visualized as follows,