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Excellent normal local domains and extensions of Krull domains ☆

William Heinzer [a], Christel Rotthaus [b], Sylvia Wiegand [c,*]

[a] Department of Mathematics, Purdue University, West Lafayette, IN 47907, United States
[b] Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, United States
[c] Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130, United States

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This paper is dedicated to Hans-Bjørn Foxby

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ABSTRACT

We consider properties of extensions of Krull domains such as flatness that involve behavior of extensions and contractions of prime ideals. Let $(R, \mathbf{m})$ be an excellent normal local domain with field of fractions K, let y be a nonzero element of $\mathbf{m}$ and let R^* denote the (y)-adic completion of R. For elements $\tau_1, \dots, \tau_s$ of yR^* that are algebraically independent over R, we construct two associated Krull domains: an intersection domain $A := K(\tau_1, \dots \tau_s) \cap R^*$ and its approximation domain B; see Setting 2.2.

If in addition R is countable with $\dim R \geq 2$, we prove that there exist elements $\tau_1, \dots, \tau_s, \dots$ as above such that, for each $s \in \mathbb{N}$, the extension $R[\tau_1, \dots, \tau_s] \hookrightarrow R^*[1/y]$ is flat; equivalently, $B = A$ and A is Noetherian. Using this result we establish the existence of a normal Noetherian local domain B such that: B dominates R; B has (y)-adic completion R^*; and B contains a height-one prime ideal $\mathbf{p}$ such that $R^*/\mathbf{p}R^*$ is not reduced. Thus B is not a Nagata domain and hence is not excellent.

We present several theorems involving the construction. These theorems yield examples where $B \subsetneq A$ and A is Noetherian while B is not Noetherian; and other examples where $B = A$ and A is not Noetherian.

1. Introduction

About twenty years ago Judy Sally gave an expository talk on the following question:

Question 1.1. What rings lie between a Noetherian integral domain S and its field of fractions $\mathcal{Q}(S)$?

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* Corresponding author.
E-mail addresses: heinzer@math.purdue.edu (W. Heinzer), rotthaus@math.msu.edu (C. Rotthaus), swiegand@math.unl.edu (S. Wiegand).

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We are inspired by work of Shreeram Abhyankar such as that in his paper [1] to ask the following related question[1]:

Question 1.2. Let I be an ideal of a Noetherian integral domain R and let R^* denote the I-adic completion of R. What rings lie between R and R^*?

A wide variety of integral domains fit the descriptions of both Questions 1.1 and 1.2. Let $(R, \mathbf{m})$ be an excellent normal local domain and let S be a polynomial ring in finitely many variables over R. In work over a number of years related to these questions, the authors have been developing techniques for constructing examples that are birational extensions of S and also subrings of an ideal-adic completion of R. Classical constructions of Noetherian integral domains with interesting properties, such as failure to be a Nagata ring, have been given by Akizuki, Schmidt, Nagata and others, [2,10,7]. We recall that a ring A is a *Nagata ring* if A is Noetherian and if the integral closure of A/P in L is finite over A/P, for every prime ideal P of A and every field L finite algebraic over the field of fractions of A/P, [6, p. 264].

We often use in our construction the completion of an excellent normal local domain $(R, \mathbf{m})$ with respect to a principal ideal yR, where y is a nonzero nonunit of R. The (y)-adic completion R^* of R may be regarded as either an inverse limit or as a homomorphic image of a formal power series ring $R[[z]]$ over R. Thus we have

$$R^* = \varprojlim_n \left(\frac{R}{y^n R} \right) = \frac{R[[z]]}{(z-y)R[[z]]}.$$

An element τ of R^* has an expression as a power series in y with coefficients in R. It is often the case that there exist elements $\tau_1, \ldots, \tau_n$ in R^* that are algebraically independent over R. An elementary cardinality argument shows this is always the case if R is countable. Assume that $\tau_1, \ldots, \tau_n$ in R^* are algebraically independent over R. By modifying τ_i by an element in R, we may assume that each $\tau_i \in yR^*$. Let $S := R[\tau_1, \ldots, \tau_n]$. Then S is both a subring of R^* and a polynomial ring in n variables over R. Although the expression for the τ_i as power series in y with coefficients in R is not unique, we use it to construct an integral domain B that is a directed union of localized polynomial rings over R.

The construction we consider associates with R and $\tau_1, \ldots, \tau_n$ the following two integral domains:

(1) an intersection domain $A := \mathcal{Q}(S) \cap R^*$, and
(2) an integral domain $B \subseteq A$ that approximates A.

The integral domain B is a directed union of localized polynomial rings in n variables over R. The rings B and A are birational extensions of the polynomial ring S and subrings of R^*. Thus they fit the descriptions of both Questions 1.1 and 1.2. For integral domains A and B obtained as above, we ask:

Questions 1.3. For a given excellent normal local domain $(R, \mathbf{m})$ and elements $\tau_1, \ldots, \tau_n$ as above, what properties do the constructed rings A and B have, and what criteria determine these properties?

Our work in this article concerning Questions 1.3 focuses primarily on the case where the base ring R is an excellent normal local domain. The intersection domain $A = \mathcal{Q}(S) \cap R^*$ may fail to be Noetherian even though R, and therefore R^*, is an excellent normal local domain. However, the intersection domain A is always a Krull domain, and the (y)-adic completion of A is R^*. Thus, in order to present an iterative procedure, in Section 2 we present many of the properties we study with the following Krull domain setting:

[1] Ram's work demonstrates the vastness of power series rings. The authors have fond memories of many pleasant conversations with him concerning power series.