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Uniform symbolic topologies and finite extensions [☆]

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This paper is dedicated to the memory and many accomplishments of Hans-Bjørn Foxby

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ABSTRACT

We study the behavior of rings with uniform symbolic topologies with respect to finite extensions.

1. Introduction

The purpose of this note is to make some observations related to the following question.

Question 1.1. *Let $(R, \mathfrak{m})$ be a complete local domain. Does there exist a positive integer b such that $P^{(bn)} \subseteq P^n$, for all prime ideals $P \subseteq R$ and all $n \geq 1$?*

Here we write $P^{(t)}$ to denote the tth symbolic power of the prime ideal P. For any Noetherian domain R, when b as above exists, we shall say that R satisfies the *uniform symbolic topology property* on prime ideals. Uniform results of this type for regular rings were first given by Ein, Lazarsfeld and Smith in [5] and by Hochster and Huneke in [7]. These results prove that if R is a regular local ring containing a field, and d is the Krull dimension of R, then $P^{((d-1)n)} \subseteq P^n$, for all prime ideals $P \subseteq R$ and all $n \geq 1$. In [12], similar uniform results were proved for large classes of isolated singularities. In general, little is known: see the introduction to [12] for further discussion about this problem. Because a complete local domain is a finite extension of a regular local ring, we were led to consider how the uniform symbolic topology property behaves with respect to finite ring extensions. Thus, Question 1.1 would have a positive answer for complete local domains containing a field if, whenever $S \subseteq R$ is a finite extension of Noetherian domains, R has the uniform

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symbolic topology property on prime ideals if S has the uniform symbolic topology property on prime ideals. Unfortunately, we are not able to show that the uniform symbolic topology property for prime ideals lifts to a finite extension, but in Section 4, we are able to show that the property lifts for a certain family of ideals in the larger extension. In Section 3, we study the easier problem of the descent of the uniform symbolic topology property in a finite extension. In Section 2, we briefly review some important results concerning the equivalence of the adic and symbolic topologies of primes ideals, and more general ideals.

Remark 1.2. With no real extra effort, one could study potentially more precise uniform estimates by introducing two integers a, b in our definition of uniformity so that one can talk about the supremum of values $\frac{b}{a}$ for which $P^{(bn)} \subseteq P^{an}$ for all primes P and all n, whenever uniform bounds exist. It should be noted that the results in [5] and [7] cannot be improved in this asymptotic sense, as shown by Bocci and Harbourne in [2]. However, since we do not obtain very precise estimates, we have elected not to present the results in this manner, but to leave it as an interesting direction to pursue.

Remark 1.3. Many of the results in this paper concern uniform topologies for primes. One might wonder why we have restricted to prime ideals as opposed to, for example, self-radical ideals. (In fact, some of our results, e.g., Theorem 4.3, do apply to self-radical ideals.) However, the problem is the following: suppose one knows a given ring R has the uniform symbolic topology property with a uniform number b, and wishes to prove that R also has uniform symbolic topologies for self-radical ideals. Let $I = Q_1 \cap \cdots \cap Q_k$ be an intersection of primes. Then $I^{(bn)} = Q_1^{(bn)} \cap \cdots \cap Q_k^{(bn)} \subset Q_1^n \cap \cdots \cap Q_k^n$. But one still must compare, in a uniform way, the intersection of powers of the Q_i with powers of their intersection. It is obvious that $(Q_1^n \cap \cdots \cap Q_k^n)^k \subset (Q_1 \cap \cdots \cap Q_k)^n$ but k is not uniform for the ring. This limits the effectiveness of some of our methods.

We use several techniques in this paper. The ideas in the paper [6] play an important role in this paper. Standard facts on Galois extensions are used. The results and ideas behind the uniform Artin–Rees theorem are crucial. We combine theorems [9, Theorem 4.13] and [10, Theorem 5.4] into the following result:

Theorem 1.4. *Let R be a reduced ring satisfying one of the following conditions:*

(1) *R is essentially of finite type over either an excellent Noetherian local ring or $\mathbb{Z}$.*
(2) *R is a ring of characteristic p, and R is module finite over R^p.*
(3) *R is an excellent Noetherian ring which is the homomorphic image of a regular ring of finite Krull dimension such that for all primes P of R, R/P has a resolution of singularities obtained by blowing up an ideal.*

Then exists a positive integer ℓ such that for all ideals I of R and all $n > \ell$, $\overline{I^n} \subset I^{n-\ell}$.

Here we are using the notation $\overline{J}$ to denote the integral closure of an ideal J.

The previous results lead to the following definition.

Definition 1.5. We say that a ring R is *acceptable* if R and all finite extension rings of R satisfy one of the following conditions: they are either essentially of finite type over an excellent Noetherian local ring or the integers, have characteristic p and are F-finite, or are an excellent Noetherian ring with infinite residue fields such that all domains which homomorphic images have a resolution of singularities obtained by blowing up an ideal. Observe that if R is as in one of the first two cases, then R is acceptable since rings of the first two types are automatically preserved under finite extensions.

For unexplained terminology, we refer the reader to the book [4].