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Duality and noncommutative planes

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ABSTRACT

We study extensions of simple modules over an associative ring A and we prove that for twosided ideals $\mathfrak{m}$ and $\mathfrak{n}$ with artinian factors the condition $\text{Ext}^1_A(A/\mathfrak{m}, A/\mathfrak{n}) \neq 0$ holds for the left A-modules $A/\mathfrak{m}$ and $A/\mathfrak{n}$ if and only if it holds for the right modules $A/\mathfrak{n}$ and $A/\mathfrak{m}$.

The methods proving this are applied to show that noncommutative models of the plane, i.e. algebras of the form $k\langle x, y\rangle/(f)$, where $f \in ([x, y])$ are noetherian only in case $(f) = ([x, y])$.

1. Introduction

For a non-commutative ring A and simple left A-modules P and Q the $\text{Ext}^1_A(P, Q)$-group has been studied intensively in several recent papers ([2], [3] and [4]). In this note we prove a sort of duality concerning simple right modules and simple left modules.

We prove that many noncommutative curves, i.e. algebras of the form $k\langle x, y\rangle/(f)$ for some $f \in k\langle x, y\rangle$, are not noetherian. This is proved by combining the method showing the duality result above with classical results.

By a similar method we also prove that for an $f \in ([x, y])$ the algebra $k\langle x, y\rangle/(f)$ is noetherian only in case $(f) = ([x, y])$.

2. Duality

In the rest of this paper A denotes an associative ring with an identity element and k an infinite field.

The main result of this section is the following.

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Theorem 2.1. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be twosided maximal ideals with artinian factors. The condition* $\mathrm{Ext}^1_A(A/\mathfrak{m}, A/\mathfrak{n}) \neq 0$ *holds for the left* A*-modules* $A/\mathfrak{m}$ *and* $A/\mathfrak{n}$ *if and only if* $\mathrm{Ext}^1_A(A/\mathfrak{n}, A/\mathfrak{m}) \neq 0$ *for the right modules* $A/\mathfrak{n}$ *and* $A/\mathfrak{m}$*.*

Proof. Suppose we have a non-zero "Ext"-group as left A-modules.

We have an exact sequence

$$0 \to \mathfrak{m} \to A \to A/\mathfrak{m} \to 0.$$

From this we get an exact sequence of right $A/\mathfrak{n}$-modules

$$\mathrm{Hom}_A(A, A/\mathfrak{n}) \xrightarrow{\phi} \mathrm{Hom}_A(\mathfrak{m}, A/\mathfrak{n}) \to \mathrm{Ext}^1_A(A/\mathfrak{m}, A/\mathfrak{n}) \to 0.$$

Hence $\mathrm{Ext}^1_A(A/\mathfrak{m}, A/\mathfrak{n}) \neq 0$ if and only if ϕ is not onto.

We claim that ϕ is onto precisely when $(\mathfrak{n} \cap \mathfrak{m}) \neq \mathfrak{n}\mathfrak{m}$.

Since $A/\mathfrak{n}$ is semisimple we have

$$\mathfrak{m}/(\mathfrak{n}\mathfrak{m}) \simeq \mathfrak{m}/(\mathfrak{n} \cap \mathfrak{m}) \oplus (\mathfrak{n} \cap \mathfrak{m})/(\mathfrak{n}\mathfrak{m}).$$

As $A/\mathfrak{n}$ is semisimple we have that an $A/\mathfrak{n}$-module M is zero exactly when $\mathrm{Hom}_A(M, A/\mathfrak{n})$ is.

Any homomorphism $\mathfrak{m} \to A/\mathfrak{n}$ vanishes on $\mathfrak{n}\mathfrak{m}$, but one that is the restriction of some homomorphism $A \to A/\mathfrak{n}$ vanishes also on $\mathfrak{n} \cap \mathfrak{m}$.

Our claim then follows by combining these observations. Thus we have proved that

$$\mathrm{Ext}^1_A(A/\mathfrak{m}, A/\mathfrak{n}) \neq 0 \text{ as left modules if and only if } \mathfrak{n} \cap \mathfrak{m} \neq \mathfrak{n}\mathfrak{m}.$$

By a completely similar argument we get that

$$\mathrm{Ext}^1_A(A/\mathfrak{n}, A/\mathfrak{m}) \neq 0 \text{ as right modules if and only if } \mathfrak{n} \cap \mathfrak{m} \neq \mathfrak{n}\mathfrak{m}. \qquad \square$$

Corollary 2.2. *From [1, Exercise 11C] we conclude that there is a link from* $\mathfrak{n}$ *to* $\mathfrak{m}$*,* $\mathfrak{n} \rightsquigarrow \mathfrak{m}$ *in case* $\mathrm{Ext}^1_A(A/\mathfrak{n}, A/\mathfrak{m}) \neq 0$ *as left* A*-modules.*

We need the next remark in Section 4 of this paper.

Remark 2.3. From [5, Corollary 6.18] we get that for a noetherian ring there is at most a countable number of maximal twosided ideals $\mathfrak{n}$ such that for a given $\mathfrak{m}$, $\mathrm{Ext}^1_A(A/\mathfrak{n}, A/\mathfrak{m}) \neq 0$ and there is only a finite number of such modules in case the corresponding simple modules are 1-dimensional.

3. Algebraic sets and models of the plane

We use the notation and the result from [4, Sections 1 and 2]. Recall that by f_0 we denote the homomorphic image of an element $f \in k\langle x, y\rangle$ in $k[x, y]$.

We wish to calculate $\mathrm{Ext}^1_A(P, Q)$ for 1-dimensional simple left A-modules P and Q. A simple method for doing so can be found in [2, Sections 4 and 5] and in [4, Section 2]. For the readers convenience we briefly recall this construction (cf. [4]).

Let $S = k\langle x_1, ..., x_m\rangle$ be the free k-algebra on m noncommuting variables. Suppose $0 \neq f \in S$ and ϕ_P is the 1-dimensional representation of S corresponding to a point $P = (a_1, ..., a_m) \in A^m_k$.