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Cohen factorizations: Weak functoriality and applications

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To Hans-Bjørn Foxby on the occasion of his sixty-fifth birthday

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ABSTRACT

We investigate Cohen factorizations of local ring homomorphisms from three perspectives. First, we prove a "weak functoriality" result for Cohen factorizations: certain morphisms of local ring homomorphisms induce morphisms of Cohen factorizations. Second, we use Cohen factorizations to study the properties of local ring homomorphisms (Gorenstein, Cohen–Macaulay, etc.) in certain commutative diagrams. Third, we use Cohen factorizations to investigate the structure of quasi-deformations of local rings, with an eye on the question of the behavior of CI-dimension in short exact sequences.

1. Introduction

Convention. Throughout this paper, the term "ring" is short for "commutative noetherian ring with identity". A local ring is "complete" when it is complete with respect to its maximal ideal. Let R, S, and T be rings.

Cohen factorizations were introduced in [9] as tools to study local ring homomorphisms. (See Section 2 for definitions and background material.) The utility of these factorizations can be seen in their many applications; see, e.g., [3,4,6–8,13,22,28]. The main point of this construction is that it allows one to study a local ring homomorphism by replacing it with a surjective one; thus, one can assume that the target is finitely generated over the source, so one can apply finite homological algebra techniques.

In Section 3 of this paper, we investigate functorial properties of Cohen factorizations. The main result of this section is the following; its proof is in 3.4. Example 3.5 shows that the separability assumptions (2) are necessary.

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Theorem A. *Consider a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccc} (R,\mathfrak{m}) & \xrightarrow{\varphi} & (S,\mathfrak{n}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ (\widetilde{R},\widetilde{\mathfrak{m}}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{S},\widetilde{\mathfrak{n}}) \end{array} \tag{A.1}$$

with the following properties:

(1) α *and* φ *have regular factorizations* $R \xrightarrow{\dot{\alpha}} R'' \xrightarrow{\alpha'} \widetilde{R}$ *and* $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$, *e.g.,* $\widetilde{R}$ *and* S *are complete,*
(2) $\widetilde{S}$ *is complete, and the field extensions* $R/\mathfrak{m} \to \widetilde{R}/\widetilde{\mathfrak{m}} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ *are separable.*

Let $S \xrightarrow{\dot{\beta}} S' \xrightarrow{\beta'} \widetilde{S}$ *and* $\widetilde{R} \xrightarrow{\dot{\widetilde{\varphi}}} \widetilde{R}' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ *be Cohen factorizations of* β *and* $\widetilde{\varphi}$. *Then there is a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccccc} R & \xrightarrow{\dot{\varphi}} & R' & \xrightarrow{\varphi'} & S \\ {\scriptstyle\dot{\alpha}}\downarrow & & \downarrow{\scriptstyle\dot{\gamma}} & & \downarrow{\scriptstyle\dot{\beta}} \\ R'' & \xrightarrow{\dot{\sigma}} & T & \xrightarrow{\sigma'} & S' \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\gamma'} & & \downarrow{\scriptstyle\beta'} \\ \widetilde{R} & \xrightarrow{\dot{\widetilde{\varphi}}} & \widetilde{R}' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S} \end{array} \tag{A.2}$$

such that the diagrams $R' \xrightarrow{\dot{\gamma}} T \xrightarrow{\gamma'} \widetilde{R}'$ *and* $R'' \xrightarrow{\dot{\sigma}} T \xrightarrow{\sigma'} S'$ *are minimal Cohen factorizations.*

We think of this as a result about functoriality of regular (e.g., Cohen) factorizations as follows. The diagram (A.1) is a morphism in the category of local ring homomorphisms. Our result provides the following commutative diagram of local ring homomorphisms where $\gamma = \gamma'\dot{\gamma}$

$$\begin{array}{ccccc} R & \xrightarrow{\dot{\varphi}} & R' & \xrightarrow{\varphi'} & S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ \widetilde{R} & \xrightarrow{\dot{\widetilde{\varphi}}} & \widetilde{R}' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S} \end{array}$$

which is a morphism in the category of regular factorizations. Of course, the operation that maps a local ring homomorphism to a regular (or Cohen) factorization is not well-defined; hence our terminology "weak functoriality".

Given a diagram (A.1) where α and β are "nice", the maps φ and $\widetilde{\varphi}$ are intimately related. This maxim is the subject of Section 4, which culminates in the proof of the next result. It is one of the applications of Cohen factorizations mentioned in the title; see 4.9 for the proof.

Theorem B. *Consider a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccc} (R,\mathfrak{m}) & \xrightarrow{\varphi} & (S,\mathfrak{n}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ (\widetilde{R},\widetilde{\mathfrak{m}}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{S},\widetilde{\mathfrak{n}}) \end{array}$$