Journal of Pure and Applied Algebra 219 (2015) 646–651

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Homological dimensions and special base changes

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A R T I C L E   I N F O

Article history:
Available online 16 May 2014

MSC:
16E10; 13D05; 13D25; 13H10

A B S T R A C T

We relate the homological behavior of an associative ring R and those of the rings R/xR and R_x when x is a regular central element in R. For left weak global dimensions we prove $\operatorname{wgldim}(R) \leq \max\{1 + \operatorname{wgldim}(R/xR), \operatorname{wgldim}(R_x)\}$ with equality if $\operatorname{wgldim}(R/xR)$ is finite. The key point is a formula for flat dimensions of R-modules: $\operatorname{fd}_R M = \max\{\operatorname{fd}_{R/xR}((R/xR) \otimes^{\mathrm{L}}_R M), \operatorname{fd}_{R_x} M_x\}$. For left noetherian R we recover results of Li, Van den Bergh and Van Oystaeyen [3] on global and projective dimensions. Similar formulae hold for injective dimensions.

1. Introduction

Throughout this paper R is a non-zero associative ring with a unit element and $x \in R$ is a central element that is neither a zero-divisor nor invertible. In particular, xR is a proper two-sided ideal and R/xR is a ring. As usual, R_x denotes the ring obtained from R by inverting the elements of the central multiplicatively closed set $\{x^n \mid n \geq 0\}$. Unless otherwise specified, rings act on their modules from the left.

The *weak global dimension* (resp., *global dimension*) of R is denoted by $\operatorname{wgldim}(R)$ (resp., $\operatorname{gldim}(R)$). It is the supremum of the flat dimensions $\operatorname{fd}_R M$ (resp., projective dimensions $\operatorname{pd}_R M$) as M ranges over all R-modules. One of our main results states:

$$\operatorname{wgldim}(R) \leq \max\bigl\{1 + \operatorname{wgldim}(R/xR), \operatorname{wgldim}(R)\bigr\}$$

with equality if $\operatorname{wgldim}(R)$ is finite. It implies a result of Li, Van den Bergh and Van Oystaeyen [3] concerning global dimensions when R is left and right noetherian.

The key ingredient in our proof is the equality

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http://dx.doi.org/10.1016/j.jpaa.2014.05.018
0022-4049/

$$\operatorname{fd}_R M = \max\{\operatorname{fd}_{R/xR}(\overline{R} \otimes^{\mathrm{L}}_R M), \operatorname{fd}_{R_x} M_x\}$$

which holds for every R-module M and $\overline{R} = R/xR$. Even over nice rings it is impossible to replace the derived tensor product $\otimes^{\mathrm{L}}_R$ with the usual tensor product:

Example 1.1. Let R be the discrete valuation domain $\mathbb{Z}_{(2)}$, M be the R-module $R/2R$ and x be the element 2 in R. Then the ring R/xR is a field, $\overline{R} \otimes_R M = M$, and $M_x = 0$. Thus, $\operatorname{fd}_R M = 1$, $\operatorname{fd}_{R/xR}(\overline{R} \otimes_R M) = 0$, and $\operatorname{fd}_{R_x} M_x = -\infty$.

The basic idea of this paper was conceived by Hans-Bjørn Foxby, more than a decade after one of the authors (S.Y.) finished his formal apprenticeship with him. The paper grew out of discussion with him. The initiators were located at different continents with several academic commitments; add Foxby's characteristic perfectionism, and it sounds not unexpected that they described the project as a kind of chess with one move per year. The authors dedicate this article to Foxby on his 65th birthday.

2. Prerequisites

In this article, we write 'R-complex' in place of 'a complex of R-modules'. Complexes are graded homologically. Thus, an R-complex M has the form

$$\cdots \to M_{\ell+1} \xrightarrow{\partial^M_{\ell+1}} M_\ell \xrightarrow{\partial^M_\ell} M_{\ell-1} \to \cdots.$$

Modules are considered to be complexes concentrated in degree zero. We write ΣM for complex with

$$(\Sigma M)_n = M_{n-1} \quad \text{and} \quad \partial^{\Sigma M} = -\partial^M.$$

The *supremum* and *infimum* of M are defined as follows:

$$\sup(M) = \sup\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(M) \neq 0\},$$
$$\inf(M) = \inf\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(M) \neq 0\},$$

with the usual conventions that one sets $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

In the rest of this section $x \in R$ is a central element. For each R-complex M multiplication by x defines an endomorphism of R-complexes

$$x_M\colon M \longrightarrow M.$$

Its mapping cone $\mathrm{C}(x_M)$ defines a long exact sequence of R-modules

$$\cdots \xrightarrow{x} \mathrm{H}_{\ell+1}(M) \longrightarrow \mathrm{H}_{\ell+1}(\mathrm{C}(x_M)) \longrightarrow \mathrm{H}_\ell(M) \xrightarrow{x} \mathrm{H}_\ell(M) \longrightarrow \cdots \tag{2.0.1}$$

Lemma 2.1. *For any R-complex M and any integer ℓ, if $\mathrm{H}_{\ell+1}(\mathrm{C}(x_M)) = 0$ and $\mathrm{H}_\ell(M_x) = 0$, then $\mathrm{H}_\ell(M) = 0$.*

Proof. Observe that $\mathrm{H}_\ell(M_x) = \mathrm{H}_\ell(M)_x$ and the map $\mathrm{H}_\ell(M) \to \mathrm{H}_\ell(M_x)$ is injective if and only if $x_{\mathrm{H}_\ell(M)}$ is injective. The result then is a direct consequence of (2.0.1). □

Lemma 2.2. *For any R-complex M, the following hold:*

(i) $\sup M = \max\{\sup \mathrm{C}(x_M) - 1, \sup M_x\}$.
(ii) $\min\{\inf \mathrm{C}(x_M), \inf M_x\} \geq \inf M \geq \min\{\inf \mathrm{C}(x_M) - 1, \inf M_x\}$.