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A note on the Matlis dual of a certain injective hull

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ABSTRACT

Let $(R, \mathfrak{m})$ denote a local ring with $E = E_R(R/\mathfrak{m})$ the injective hull of the residue field. Let $\mathfrak{p} \in \operatorname{Spec} R$ denote a prime ideal with $\dim R/\mathfrak{p} = 1$, and let $E_R(R/\mathfrak{p})$ be the injective hull of $R/\mathfrak{p}$. As the main result we prove that the Matlis dual $\operatorname{Hom}_R(E_R(R/\mathfrak{p}), E)$ is isomorphic to $\widehat{R_\mathfrak{p}}$, the completion of $R_\mathfrak{p}$, if and only if $R/\mathfrak{p}$ is complete. In the case of R a one dimensional domain there is a complete description of $Q \otimes_R \hat{R}$ in terms of the completion $\hat{R}$.

1. Introduction

Let R denote a commutative Noetherian ring. For injective R-modules I, J it is well known that $\operatorname{Hom}_R(I, J)$ is a flat R-module. In order to understand them the first case of interest is when I, J are indecomposable (as follows by Matlis' Structure Theory (see e.g. [7] or [3])).

Let $(R, \mathfrak{m})$ denote a local ring with the injective hull $E = E_R(R/\mathfrak{m})$ of the residue field $k = R/\mathfrak{m}$. In this situation it comes down to understanding the Matlis dual $\operatorname{Hom}_R(I, E)$ of an injective R-module, in particular for $I = E_R(R/\mathfrak{p})$, the injective hull of $R/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} R$. It was shown (see [3, 3.3.14] and [3, 3.4.1 (7)]) that

$$\operatorname{Hom}_R\big(E_R(R/\mathfrak{p}), E\big) \simeq \operatorname{Hom}_R\big(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{p})^{\mu_\mathfrak{p}}\big) \simeq \widehat{R_\mathfrak{p}^{\mu_\mathfrak{p}}}.$$

Moreover it follows (see [3, 3.3.10]) that

$$\mu_\mathfrak{p} = \dim_{k(\mathfrak{p})} \operatorname{Hom}_R\big(k(\mathfrak{p}), E\big).$$

Therefore $\operatorname{Hom}_R(E_R(R/\mathfrak{p}), E)$ is the completion of a free $R_\mathfrak{p}$-module of rank $\mu_\mathfrak{p}$.

Here we shall prove – as the main result of the paper – the following result on the Matlis dual of a certain $E_R(R/\mathfrak{p})$.

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Theorem 1.1. *Let $(R, \mathfrak{m})$ denote a local ring. Let $\mathfrak{p}$ denote a one dimensional prime ideal. Then $\operatorname{Hom}_R(E_R(R/\mathfrak{p}), E) \simeq \widehat{R_\mathfrak{p}}$ (i.e. it is the completion of a free $R_\mathfrak{p}$-module of rank one) if and only if $R/\mathfrak{p}$ is complete.*

Let $\mathfrak{p}$ denote a one dimensional prime ideal in a local ring $(R, \mathfrak{m})$. The equality $\mu_\mathfrak{p} = 1$ was proved in [4], resp. in [5], in the case of R a complete Gorenstein domain, resp. in the case of R a complete Cohen–Macaulay domain. The proofs are based on the use of the dualizing module of a complete Cohen–Macaulay domain. Note that the dualizing module is isomorphic to R in the case of a complete Gorenstein domain.

Here we use as a basic ingredient Matlis duality and – as a main step – the reduction to the case of $\dim R = 1$ suggested by one of the reviewers. In the case of a one dimensional domain there is a complete description of $\operatorname{Hom}_R(E_R(R), E)$ and $Q \otimes_R \hat{R}$ in terms of the completion $\hat{R}$ (see Theorem 2.5 for the precise formulation).

2. Proofs

In the following $(R, \mathfrak{m})$ denotes always a local ring with $E = E_R(R/\mathfrak{m})$ the injective hull of the residue field $R/\mathfrak{m}$. Then $D_R(\cdot) = \operatorname{Hom}_R(\cdot, E)$ denotes the Matlis duality functor.

Remark 2.1. (A) Let X be an arbitrary R-module. There is a natural homomorphism

$$X \to D\big(D(X)\big)$$

that is always injective. If $(R, \mathfrak{m})$ is complete it is an isomorphism whenever X is an Artinian R-module, resp. a finitely generated R-module (see [7, p. 528] and [7, Corollary 4.3]). Moreover it follows that the map is an isomorphism if and only if there is a finitely generated R-submodule $Y \subset X$ such that X/Y is an Artinian R-module. For the proof we refer to [8] and also to [1] for a generalization.

(B) Let M denote a finitely generated R-module. Then there is a natural isomorphism $M \otimes_R \hat{R} \simeq D(D(M))$. That is, M is Matlis reflexive if an only if it is complete.

(C) Let X denote an R-module with $\operatorname{Supp}_R X \subset \{\mathfrak{m}\}$. Then M admits the structure of an $\hat{R}$-module compatible with its R-module structure such that $X \otimes_R \hat{R} \to X$ is an isomorphism (see e.g. [8, (2.1)]). Let M denote an R-module and N an $\hat{R}$ module. Then $\operatorname{Ext}^i_R(M, N), i \in \mathbb{Z}$, has the structure of an $\hat{R}$-module. Moreover, here are natural isomorphisms

$$\operatorname{Ext}^i_R(M, N) \simeq \operatorname{Ext}^i_{\hat{R}}(M \otimes_R \hat{R}, N)$$

for all $i \in \mathbb{Z}$ since $\hat{R}$ is a flat R-module.

As a technical tool we shall need the short exact sequence of the following trivial lemma.

Lemma 2.2. *Let $(R, \mathfrak{m})$ denote a one dimensional domain. Then there is a short exact sequence*

$$0 \to R \to Q \to H^1_\mathfrak{m}(R) \to 0$$

where $Q = \mathbb{Q}(R)$ denotes the quotient field of R.

Proof. We start with the following short exact sequence $0 \to R \to Q \to Q/R \to 0$. The long exact local cohomology sequence provides an isomorphism $Q/R \simeq H^1_\mathfrak{m}(R)$. To this end recall that $H^i_\mathfrak{m}(Q) = 0$ for all $i \in \mathbb{Z}$. This proves the statement. □