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Transitive permutation groups where nontrivial elements have at most two fixed points

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ABSTRACT

Motivated by a question on Riemann surfaces, we consider permutation groups that act nonregularly, such that every nontrivial element has at most two fixed points. We describe the permutation groups with these properties and give a complete, detailed classification when the group is simple.

1. Introduction

From its inception the theory of permutation groups has been concerned with the question of how fixed point sets of elements influence the structure of a permutation group. For example a celebrated result by Jordan is the fact that a primitive permutation group which contains a transposition is the full symmetric group. The special case where the permutation group has degree 5 was used by Galois to prove the unsolvability of the general quintic. Jordan's result was strengthened and generalized in several ways. For example the maximal subgroups of S_n which contain transpositions are either intransitive of the form $S_k \times S_{n-k}$ or imprimitive of the form $S_k \wr S_{n/k}$. Other results concern primitive permutation groups in which some nonidentity element fixes many points. For example in [16] one can find the explicit list of exceptions to the statement that a primitive permutation group where some nonidentity element fixes at least half the points of the permutation domain contains the full alternating group.

We would like to determine the group theoretic structure of a transitive permutation group where all nontrivial elements fix at most a bounded number of, say k, points. Of course the case $k = 0$ is when the action of the group on its permutation domain is regular. The case where $k = 1$ is the situation that Frobenius characterized; i.e. Frobenius groups. Motivated by an application to the theory of compact Riemann surfaces, see Corollary 1.5, we investigate the case $k = 2$. Although we do not impose any hypothesis on primitivity or higher transitivity, we would like to mention the related work of Zassenhaus and Suzuki, see for example [26,25] and Theorem 2.9 in [22]. From this point of view the series of groups that we encounter are no

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surprise. In fact our results show that groups satisfying our more general hypotheses can have arbitrarily large permutation rank. For details, see the remarks at the end of this article.

From now on we operate under the following

Hypothesis 1.1. *Suppose that G is a finite, transitive, nonregular permutation group with permutation domain Ω. Suppose further that $|\Omega| \geq 4$ and that every element $g \in G^{\#}$ has at most two fixed points.*

For simple groups we prove

Theorem 1.2. *Suppose that (G, Ω) satisfies Hypothesis 1.1 and that G is simple. Then either G is isomorphic to $PSL_3(4)$ or there exists a prime power q such that G is isomorphic to $PSL_2(q)$ or to $Sz(q)$.*

We will show in the last section that there are no quasisimple, nonsimple examples. For the almost simple groups we have

Theorem 1.3. *Suppose that G is almost simple, but not simple. Let $E := F^*(G)$ and suppose that (G, Ω) satisfies Hypothesis 1.1. Then there exists a prime power q such that $E \cong PSL_2(q)$ and one of the following holds:*

(a) $G \cong PGL_2(q)$.
(b) q is a power of 2, moreover $|G : E|$ is prime, and there exists an element $g \in G \backslash E$ such that g induces a field automorphism on E and $C_E(g) \cong PSL_2(2)$.
(c) q is odd, $|G : E| = 2$ and the elements of $G \setminus E$ act as diagonal-field automorphisms of order 2. This includes the case where $E \cong PSL_2(9) \cong \mathcal{A}_6$ and $G \cong M_{10}$.

Our most general result is

Theorem 1.4. *Suppose that (G, Ω) satisfies Hypothesis 1.1. Then one of the following holds:*

(1) G has a subgroup of index at most 2 that is a Frobenius group.
(2) $|Z(G)| = 2$ and $G/Z(G)$ is a Frobenius group.
(3) The point stabilizers are metacyclic of odd order. If H is a nontrivial two point stabilizer, then $|N_G(H) : H| = 2$, G is solvable and H or $N_G(H)$ has a normal complement K in G such that K is nilpotent and $(|K|, |H|) = 1$.
(4) The point stabilizers are metacyclic of odd order. Moreover G has normal subgroups N, M such that $N < M < G$, N is nilpotent, M/N is simple and isomorphic to $PSL_2(q)$, to $Sz(q)$ or to $PSL_3(4)$, and G/M is metacyclic of odd order.
(5) The point stabilizers have twice odd order and G has a subgroup M of index 2 such that either (3) or (4) holds for M or M acts regularly on Ω.
(6) The point stabilizers have even order and G has a normal subgroup N of odd order such that $O^{2'}(G)/N$ is either a dihedral or semidihedral 2-group or there exists a prime power q such that it is isomorphic to $Sz(q)$ or to a subgroup of $P\Gamma L_2(q)$ that contains $PSL_2(q)$.

In part our result was motivated by its application to the theory of Riemann surfaces. A Weierstrass point of a compact Riemann surface X of genus $g > 1$ is a point $x \in X$ such that there exists a holomorphic function which has a pole of order at most g at x and is holomorphic on $X \setminus \{x\}$. Apart from their analytic significance Weierstrass points also influence the structure of the automorphism group. For example, by considering the action on the Weierstrass points, Schwarz showed that the automorphism group of a compact Riemann surface X of genus $g > 1$ is finite. (See for example page 258 in [9].) Going in the opposite direction Schoeneberg showed that if an automorphism of a compact Riemann surface of genus $g \geq 2$ fixes at least five