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Symbolic generic initial systems of star configurations

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ABSTRACT

The purpose of this note is to describe limiting shapes of symbolic generic initial systems of star configurations in projective spaces $\mathbb{P}^n$ over a field $\mathbb{K}$ of characteristic 0.

1. Introduction

In recent years there has been increasing interest in asymptotic invariants attached to graded families of ideals, see for example [18, Section 2.4.B]. For a homogeneous ideal I, Mayes introduces in [20] *symbolic generic initial systems* $\{\mathrm{gin}(I^{(m)})\}_m$. Here $\mathrm{gin}(J)$ denotes the reverse lexicographic generic initial ideal of a homogeneous ideal J and $J^{(m)}$ denotes the mth symbolic power of J. For a monomial ideal J one defines its Newton polytope $P(J)$. Mayes studies the *limiting shape* associated with I as the asymptotic Newton polytope

$$\Delta(I) = \bigcup_{m=1}^{\infty} \frac{1}{m} P\big(\mathrm{gin}\big(I^{(m)}\big)\big)$$

for *generic* points in $\mathbb{P}^2$ under assumption of the Segre–Harbourne–Gimigliano–Hirschowitz Conjecture (see [3] for recent account on this conjecture). In this note we study limiting shapes for star configurations in projective spaces of arbitrary dimension. Star configurations have received much attention recently, partly because they are a nice source of interesting and computable examples, see for example the nice survey [12]. Our main result here is the following theorem.

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Theorem 1.1. *Let $I_{n,s,n}$ be the ideal of points defined as n-fold intersection points of $s \geqslant n$ general hyperplanes in $\mathbb{P}^n$. Then the complement of $\Delta(I_{n,s,n})$, $\Gamma(I_{n,s,n}) = \overline{(\mathbb{R}_{\geqslant 0})^n \setminus \Delta(I_{n,s,n})}$, is the simplex in $\mathbb{R}^n$ with vertices at the origin and the points $A_1, \dots, A_n$, where*

$$A_i = \left(\underbrace{0, \dots, 0}_{i-1}, \frac{s-(i-1)}{n-(i-1)}, \underbrace{0, \dots, 0}_{n-i}\right).$$

More generally, let $I_{c,s,n}$ be the ideal of the union of all linear subspaces of codimension c in $\mathbb{P}^n$ cut out by c out of s general hyperplanes in $\mathbb{P}^n$. Then

$$\Gamma(I_{c,s,n}) = \Gamma(I_{c,s,c}) \times \mathbb{R}^{n-c}_{\geqslant 0}.$$

That is, the coconvex body $\Gamma(I_{c,s,n})$ associated with $I_{c,s,n}$ is the cylinder over the coconvex body associated with $I_{c,s,c}$ restricted to the positive octant in $\mathbb{R}^n$.

2. Generic initial ideals

Let $S(n) = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ of characteristic 0. Let $\succ$ be the reverse lexicographic order on monomials in $S(n)$. Recall that $\succ$ is a total ordering defined as

$$x_1^{p_1} \cdot \dots \cdot x_n^{p_n} \succ x_1^{q_1} \cdot \dots \cdot x_n^{q_n}$$

if and only if, $\sum_{i=1}^n p_i > \sum_{i=1}^n q_i$ or $\sum p_i = \sum q_i$ and there exists an index k such that $p_\ell = q_\ell$ for all $\ell > k$ and $p_k < q_k$.

For a homogeneous ideal $I \subset S(n)$, its *initial ideal* $\mathrm{in}(I)$ is the ideal generated by leading terms of all elements of I. Recall that the *leading term* $\mathrm{in}(f)$ of a polynomial $f \in S$ is the greatest (with respect to the fixed order, here $\succ$) monomial summand of f. Initial ideals are of interest because they share many properties with the original ideals whereas they are easier to handle computationally, see for example [15].

Even better behaved are *generic initial ideals*. We describe now briefly how they are defined following Galligo [11] and Green [13]. To begin with, recall that $\mathrm{GL}(n, \mathbb{K})$ acts on $S(n)$ by the change of coordinates. The Borel subgroup $\mathbb{T}$ of $\mathrm{GL}(n, \mathbb{K})$ consists of upper triangular matrices

$$\mathbb{T} = \{A \in \mathrm{GL}(n, \mathbb{K}) : a_{ij} = 0 \text{ for all } j < i\}.$$

Theorem 1.27 of Galligo [13] assures that for a homogeneous ideal I and a generic choice of coordinates g, the initial ideal $\mathrm{in}(g(I))$ is $\mathbb{T}$-fixed. When g is such a change of coordinates, we write $\mathrm{gin}(I)$ for $\mathrm{in}(g(I))$. It follows from the same theorem that $\mathrm{gin}(I)$ is well defined, in particular uniquely determined by I. Generic initial ideals carry even more information on original ideal I than arbitrary initial ideals do. For example the *satiety* $\mathrm{sat}(I)$ of a homogeneous ideal I is the least integer m such that $I_d = I_d^{\mathrm{sat}}$ for all $d \geqslant m$ (here I^{sat} denotes the *saturation* of I). It is

$$\mathrm{sat}(I) = \mathrm{sat}(\mathrm{gin}(I))$$

by [13, Theorem 2.24], whereas the equality may fail for $\mathrm{in}(I)$. Similarly, the regularity $\mathrm{reg}(I)$ of I is equal to the regularity $\mathrm{reg}(\mathrm{gin}(I))$ of $\mathrm{gin}(I)$ by [13, Theorem 2.27]. The regularity of $\mathrm{in}(I)$ might be in general higher than that of I.