Journal of Pure and Applied Algebra 219 (2015) 1103–1121

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Components of Springer fibers associated to closed orbits for the symmetric pairs $(Sp(2n), Sp(2p) \times Sp(2q))$ and $(SO(2n), GL(n))$, II

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ARTICLE INFO

Article history:
Received 2 June 2012
Received in revised form 9 April 2014
Available online 20 May 2014
Communicated by A. Knutson

ABSTRACT

This is the second of two articles that consider the pairs of complex reductive groups $(G, K) = (Sp(2n), Sp(2p) \times Sp(2q))$ and $(SO(2n), GL(n))$ and components of Springer fibers associated to closed K-orbits in the flag variety of G. In the first an algorithm is given to compute the associated variety of any discrete series representation of $G_{\mathbf{R}} = Sp(p, q)$ and $SO^*(2n)$ and to concretely describe the corresponding component of a Springer fiber. These results are used here to compute associated cycles of discrete series representations. For each Harish-Chandra cell containing a discrete series representation, a particular discrete series representation is identified for which the structure of the component is sufficiently simple that the multiplicity in the associated cycle can be calculated. Coherent continuation is then applied to compute associated cycles of all representations in such a cell.

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0. Introduction

In this article we consider the pairs (G, K) of complex groups

$$\begin{aligned} &\big(Sp(2n), Sp(2p) \times Sp(2q)\big), \quad n = p + q, \\ &\big(SO(2n), GL(n)\big), \end{aligned} \tag{1}$$

which are referred to as type C and type D, and the corresponding real forms

$$G_{\mathbf{R}} = Sp(p, q),\ SO^*(2n). \tag{2}$$

A method is given to compute associated cycles of discrete series representations of $G_{\mathbf{R}}$. The method in fact computes the associated cycle of any representation in a Harish-Chandra cell which contains a discrete

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http://dx.doi.org/10.1016/j.jpaa.2014.05.038
0022-4049/Published by Elsevier B.V.

series representation. In each such cell we find a discrete series representation X' for which a relatively elementary argument gives us the associated cycle. We show that the corresponding cell representation is generated by X'. Then the general theory of characteristic cycles, Springer representations and coherent continuation applies to show that the computation of $AC(X')$ gives the associated cycle of any irreducible Harish-Chandra module in the cell of X'. The computation is in the form of an algorithm, which will be outlined below. An important part of the algorithm is based on the results and methods of [4].

Before describing the algorithm we establish some notations. The subgroup K is the fixed point group of an involution Θ of G. The (complexified) Cartan decomposition of the Lie algebra of G (for θ, the differential of Θ) is written as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is contained in $\mathfrak{k}$.

The principal geometric objects considered are the flag variety $\mathfrak{B}$ for G and the nilpotent cone $\mathcal{N}$ of the Lie algebra $\mathfrak{g}$. We also consider $\mathcal{N}_\theta := \mathcal{N} \cap \mathfrak{p}$. The moment map of the cotangent bundle to $\mathfrak{B}$ is denoted by $\mu : T^*\mathfrak{B} \to \mathcal{N}$. If $\mathfrak{Q} \subset \mathfrak{B}$ is a K-orbit, then the restriction of μ to the conormal bundle maps to $\mathcal{N}_\theta$. The fibers of μ play an important role and are referred to as Springer fibers; we often use the common notation $\mathfrak{B}^f$ for $\mu^{-1}(f)$. A standard fact is that when $\mu(\overline{T^*_{\mathfrak{Q}}\mathfrak{B}}) = \overline{K \cdot f}$, then $\mathfrak{B}^f \cap \overline{T^*_{\mathfrak{Q}}\mathfrak{B}}$ is a union of irreducible components of $\mathfrak{B}^f$. One easily sees that $\mathfrak{B}^f$ may be identified with the set of Borel subalgebras that contain f, thus may be identified with a subvariety of $\mathfrak{B}$.

Let X be a discrete series representation having the same infinitesimal character as the trivial representation (infinitesimal character equal to ρ). Then X is associated to a closed K-orbit $\mathfrak{Q} \subset \mathfrak{B}$, its support in the Beilinson–Bernstein description of irreducible Harish-Chandra modules. There is a coherent family $\{X_\lambda\}_{\lambda\in\Lambda}$, Λ the integral lattice in $\mathfrak{h}^*$, so that $X = X_\rho$. The associated cycle of X_λ, $\lambda \in \Lambda^+$, is

$$AC(X_\lambda) = m_{\mathfrak{Q}}(\lambda)\overline{\mathcal{O}}$$

where $\mu(T^*_{\mathfrak{Q}}\mathfrak{B}) = \overline{\mathcal{O}}$, $\mathcal{O} = K \cdot f \subset \mathcal{N}_\theta$. The definition of the associated cycle of a Harish-Chandra module is given in Section 2 of [19]. A theorem of J.-T. Chang [6, Cor. 2.9] states that the multiplicity $m_{\mathfrak{Q}}(\lambda)$ of $\overline{\mathcal{O}}$ in $AC(X_\lambda)$ is given by

$$m_{\mathfrak{Q}}(\lambda) = \dim\big(H^0\big(\mu^{-1}(f) \cap T^*_{\mathfrak{Q}}\mathfrak{B}, \mathcal{O}(\lambda + \rho - 2\rho_c)\big)\big), \tag{3}$$

for some invertible sheaf $\mathcal{O}(\lambda+\rho-2\rho_c)$. For the groups $U(p,q)$, $Sp(2n,\mathbf{R})$ and $O(p,q)$ algorithms are given in [2] and [3] to compute $AC(X_\lambda)$. The procedure there is to begin with a closed K-orbit $\mathfrak{Q}$ in $\mathfrak{B}$ (which corresponds to a positive system of roots), then construct a nice 'generic' element $f \in \mathcal{N}_\theta$ in terms of root vectors. It turns out that $\mu^{-1}(f) \cap T^*_{\mathfrak{Q}}\mathfrak{B}$, a component of the Springer fiber (or several components), has a particularly nice form and the Borel–Weil theorem can be applied to compute $m_{\mathfrak{Q}}(\lambda)$. For the real forms (2) considered in the present article, $\mu^{-1}(f) \cap T^*_{\mathfrak{Q}}\mathfrak{B}$ does not seem to have a nice form for every closed orbit $\mathfrak{Q}$. Therefore $m_{\mathfrak{Q}}(\lambda)$ cannot always be computed directly using (3).

The algorithm presented here to compute $AC(X_\lambda)$ goes as follows. As for the other classical groups, begin by finding the (nice) 'generic' element f. This is done in Section 2 of [4]. The next step is to replace $\mathfrak{Q} = \text{support}(X)$ by another closed K-orbit $\mathfrak{Q}' \subset \mathfrak{B}$ for which two H $\mu(T^*_{\mathfrak{Q}'}\mathfrak{B}) = \mu(T^*_{\mathfrak{Q}}\mathfrak{B})$ and $m_{\mathfrak{Q}'}(\lambda)$ can be computed in an elementary way using (3) and the Borel–Weil theorem. The point is that, with $\mathfrak{Q}'$ properly chosen, $\mu^{-1}(f) \cap T^*_{\mathfrak{Q}'}\mathfrak{B}$ has a very simple form (often homogeneous). This is carried out in Section 3.

The final step is to compute $m_{\mathfrak{Q}}(\lambda)$, now that $m_{\mathfrak{Q}'}(\lambda)$ is known. Suppose that X' is the discrete series representation of trivial infinitesimal character with $\mathfrak{Q}' = \text{support}(X')$ and $\{X'_\lambda\}_{\lambda\in\Lambda}$ is the coherent family with $X' = X'_\rho$. From the algorithm to compute $\mathfrak{Q}'$ from $\mathfrak{Q}$ one sees (Section 3 and [4]) that X and X' have the same associated variety. It follows from Theorems 6 and 10 of [11] that X and X' are in the same Harish-Chandra cell $\mathcal{C}$.

In Section 2 we show that X' (indeed any discrete series representation, for our $G_{\mathbf{R}}$) generates the cell representation $V_{\mathcal{C}}$ as $\mathbf{Q}[W]$-module, that is, $V_{\mathcal{C}} = \mathbf{Q}[W] \cdot X'$. It follows from the W-equivariance of the map sending a representation to its multiplicity polynomial, that