Journal of Pure and Applied Algebra 219 (2015) 1175–1210

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

A new model for pro-categories

Ilan Barnea [a,*,1], Tomer M. Schlank [b,2]

[a] *Department of Mathematics, University of Muenster, Nordrhein-Westfalen, Germany*
[b] *Department of Mathematics, Massachusetts Institute of Technology, MA, USA*

ARTICLE INFO

Article history:
Received 28 May 2013
Received in revised form 1 May 2014
Available online 13 June 2014
Communicated by C.A. Weibel

ABSTRACT

In this paper we present a new way to construct the pro-category of a category. This new model is very convenient to work with in certain situations. We present a few applications of this new model, the most important of which solves an open problem of Isaksen [7] concerning the existence of functorial factorizations in what is known as the strict model structure on a pro-category. Additionally we explain and correct an error in one of the standard references on pro-categories.

1. Introduction

Pro-categories introduced by Grothendieck [13] have found many applications over the years in fields such as algebraic geometry [1], shape theory [9] and more. Generally speaking, given a category $\mathcal{C}$ one can think of $\mathrm{Pro}(\mathcal{C})$ as the category of "inverse systems" in $\mathcal{C}$. When $\mathcal{C}$ has finite limits $\mathrm{Pro}(\mathcal{C})$ can be shown to be equivalent to the category of left exact functors from $\mathcal{C}$ to the category of Sets. While the last model has some functorial advantages, the model of $\mathrm{Pro}(\mathcal{C})$ as inverse systems is very concrete and pictorial. In this paper we suggest a new model for $\mathrm{Pro}(\mathcal{C})$ which we shall denote by $\overline{\mathrm{Pro}}(\mathcal{C})$. We think of $\overline{\mathrm{Pro}}(\mathcal{C})$ as a model in which one considers only inverse systems indexed by cofinite directed posets of infinite height. The big advantage of such indexing is that it is very susceptible to proofs by induction. The category $\overline{\mathrm{Pro}}(\mathcal{C})$ is the homotopy category of a very natural 2-category $\widetilde{\mathrm{Pro}}(\mathcal{C})$ which makes working with pro-categories quite natural.

Specifically, we use the new model to prove a few propositions concerning factorizations of morphisms in pro-categories. These will later be used to deduce certain facts about model structures on pro-categories. The most important conclusion of this paper will be solving an open problem of Isaksen [7] concerning

* Corresponding author.
E-mail addresses: ilanbarnea770@gmail.com (I. Barnea), schlank@math.mit.edu (T.M. Schlank).
[1] The first author is supported by the Alexander von Humboldt Professorship of Michael Weiss of the University of Muenster.
[2] The second author is supported by the Simons fellowship in the Department of Mathematics of the Massachusetts Institute of Technology.

http://dx.doi.org/10.1016/j.jpaa.2014.06.001
0022-4049/

the existence of functorial factorizations in what is known as the strict model structure on a pro-category. In order to state our results more accurately we give some definitions in a rather brief way. For a more detailed account see Section 2.

First recall that the category $\mathrm{Pro}(\mathcal{C})$ has as objects all diagrams in $\mathcal{C}$ of the form $I \to \mathcal{C}$ such that I is small and directed (see Definition 2.1). The morphisms are defined by the formula:

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, Y) := \lim_s \operatorname*{colim}_t \mathrm{Hom}_{\mathcal{C}}(X_t, Y_s).$$

Composition of morphisms is defined in the obvious way.

Note that not every map in $\mathrm{Pro}(\mathcal{C})$ is a natural transformation (the source and target need not even have the same indexing category). However, every natural transformation between objects in $\mathrm{Pro}(\mathcal{C})$ having the same indexing category induces a morphism in $\mathrm{Pro}(\mathcal{C})$ between these objects, in a rather obvious way.

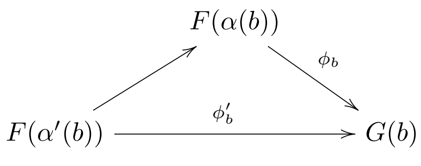
Let M be a class of morphisms in $\mathcal{C}$. We denote by $Lw^{\cong}(M)$ the class of morphisms in $\mathrm{Pro}(\mathcal{C})$ that are **isomorphic** to a morphism that comes from a natural transformation which is a levelwise M-map.

If T is a partially ordered set, then we view T as a category which has a single morphism $u \to v$ iff $u \geq v$. A cofinite poset is a poset T such that for every x in T the set $T_x := \{z \in T \mid z \leq x\}$ is finite.

Suppose now that $\mathcal{C}$ has finite limits. Let T be a small cofinite poset and $F : X \to Y$ a morphism in $\mathcal{C}^T$. Then F will be called a special M-map, if the natural map $X_t \to Y_t \times_{\lim_{s<t} Y_s} \lim_{s<t} X_s$ is in M, for every t in T. We denote by $Sp^{\cong}(M)$ the class of morphisms in $\mathrm{Pro}(\mathcal{C})$ that are **isomorphic** to a morphism that comes from a (natural transformation which is a) special M-map.

We now define the 2-category $\widetilde{\mathrm{Pro}}(\mathcal{C})$. A (strict) 2-category is a category enriched in categories. More particularly $\widetilde{\mathrm{Pro}}(\mathcal{C})$ is a category enriched in posets. Since a poset can be considered as a 1-category we indeed get a structure of a 2-category on $\widetilde{\mathrm{Pro}}(\mathcal{C})$. Let A be a cofinite directed set. We will say that A has infinite height if for every a in A there exists a' in A such that $a < a'$. An object of the 2-category $\widetilde{\mathrm{Pro}}(\mathcal{C})$ is a diagram $F : A \to \mathcal{C}$, such that A is a cofinite directed set of infinite height. If $F : A \to \mathcal{C}$ and $G : B \to \mathcal{C}$ are objects in $\widetilde{\mathrm{Pro}}(\mathcal{C})$, a 1-morphism f from F to G is a defined to be a pair $f = (\alpha_f, \phi_f)$, such that $\alpha_f : B \to A$ is a strictly increasing function, and $\phi_f : \alpha_f^* F = F \circ \alpha_f \to G$ is a natural transformation.

Given two strictly increasing maps $\alpha, \alpha' : B \to A$ we write $\alpha' \geq \alpha$ if for every b in B we have $\alpha'(b) \geq \alpha(b)$. Now we define the partial order on the set of 1-morphisms from F to G. We set $(\alpha', \phi') \geq (\alpha, \phi)$ iff $\alpha' \geq \alpha$ and for every b in B the following diagram commutes:



(the arrow $F(\alpha'(b)) \to F(\alpha(b))$ is of course the one induced by the unique morphism $\alpha'(b) \to \alpha(b)$ in A).

Composition of 1-morphisms in $\widetilde{\mathrm{Pro}}(\mathcal{C})$ is defined by the formula:

$$(\beta, \psi) \circ (\alpha, \phi) = (\alpha \circ \beta, \psi \circ \phi_\beta).$$

We define $\overline{\mathrm{Pro}}(\mathcal{C})$ to be the homotopy category of $\widetilde{\mathrm{Pro}}(\mathcal{C})$, that is, the one obtained by identifying every couple of 1-morphisms with a 2-morphism between them. Namely, a morphism between F and G in $\overline{\mathrm{Pro}}(\mathcal{C})$ is a connected component of the poset $Mor_{\widetilde{\mathrm{Pro}}(\mathcal{C})}(F, G)$. We will show (see Corollary 3.7) that every such connected component is a directed poset. Given a 1-morphism $f = (\alpha_f, \phi_f)$ in $\widetilde{\mathrm{Pro}}(\mathcal{C})$ we denote by $[f] = [\alpha_f, \phi_f]$ the corresponding morphism in $\overline{\mathrm{Pro}}(\mathcal{C})$.