Journal of Pure and Applied Algebra 219 (2015) 1211–1235

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Cyclic structures and the topos of simplicial sets

Alain Connes [a,b,c,*], Caterina Consani [d,1]

[a] Collège de France, 3 rue d'Ulm, Paris F-75005, France
[b] I.H.E.S., France
[c] Ohio State University, USA
[d] Department of Mathematics, The Johns Hopkins University, Baltimore, MD 21218, USA

A R T I C L E I N F O

Article history:
Received 30 September 2013
Received in revised form 14 April 2014
Available online 3 July 2014
Communicated by C.A. Weibel

MSC:
19D55; 18B25; 03G30

A B S T R A C T

Given a point p of the topos $\hat{\Delta}$ of simplicial sets and the corresponding flat covariant functor $\mathfrak{F} : \Delta \longrightarrow \mathfrak{Sets}$, we determine the extensions of $\mathfrak{F}$ to the cyclic category $\Lambda \supset \Delta$. We show that to each such cyclic structure on a point p of $\hat{\Delta}$ corresponds a group G_p, that such groups can be noncommutative and that each G_p is described as the quotient of a left-ordered group by the subgroup generated by a central element. Moreover for any cyclic set X the fiber (or geometric realization) of the underlying simplicial set of X at p inherits canonically the structure of a G_p-space. This gives a far reaching generalization of the well-known circle action on the geometric realization of cyclic sets.

1. Introduction

This paper aims to illustrate the unifying power of the notion of topos, due to Grothendieck, in the context of cyclic homology. Our main motivation originates from the recent discovery (*cf.* [5]) of the role of cyclic homology of schemes for the cohomological interpretation of the archimedean local factors of L-functions of arithmetic varieties. This result raises naturally the question of a conceptual interpretation of cyclic homology of schemes. In [3] it was shown that cyclic cohomology can be interpreted as Ext-functor in the category of cyclic modules. These modules are defined as contravariant functors from the (small) cyclic category Λ to the category of abelian groups. This development brings at the forefront the crucial role played by the cyclic category as an extension of the simplicial category Δ by finite cyclic groups. Moreover, in [3] it was also shown that the classifying space $B\Lambda$ is equal to the classifying space of the topological group $U(1)$, and later it was discovered (*cf.* [2,8,9]) that the geometric realization of the simplicial set underlying a cyclic set (this latter understood as a contravariant functor $\Lambda \longrightarrow \mathfrak{Sets}$ to the category of sets) inherits naturally

* Corresponding author at: Collège de France, 3 rue d'Ulm, Paris F-75005, France.
E-mail addresses: alain@connes.org (A. Connes), kc@math.jhu.edu (C. Consani).
[1] The second author is partially supported by the NSF grant DMS 1069218 and would like to thank the Collège de France for some financial support.

http://dx.doi.org/10.1016/j.jpaa.2014.06.002
0022-4049/

an action of $U(1)$. The equality $B\Lambda = BU(1)$ then leads to a deep analogy between cyclic cohomology and $U(1)$-equivariant cohomology.

In this article we show that one obtains a conceptual understanding of the above $U(1)$-action on the geometric realization of cyclic sets by extending that result to the framework of topos theory. The transition from a small category to its classifying space produces in general a substantial loss of information: the classifying space of Δ is, for instance, a singleton. It is exactly at this point that the implementation of topos theory turns out to be useful to provide the correct environment that encloses both schemes and small categories while also furnishing the tools for the development of cohomology. In the words of Grothendieck:

L'idée du topos englobe, dans une intuition topologique commune, aussi bien les traditionnels espaces (topologiques), incarnant le monde de la grandeur continue, que les (soi-disant) « espaces » (ou « variétés ») des géomètres algébristes abstraits impénitents, ainsi que d'innombrables autres types de structures, qui jusque-là avaient semblé rivées irrémédiablement au « monde arithmétique » des agrégats « discontinus » ou « discrets ».

The category $\mathrm{Sh}(X)$ of sheaves of sets on a topological space X is a topos that captures all the relevant information on X. For a small category $\mathcal{C}$, the associated category $\hat{\mathcal{C}} = \mathfrak{Sets}^{\mathcal{C}^{\mathrm{op}}}$ of contravariant functors $\mathcal{C} \longrightarrow \mathfrak{Sets}$ is a topos. What is more, the notion of point is meaningful for any topos $\mathcal{T}$: a point is simply a geometric morphism $f : \mathfrak{Sets} \longrightarrow \mathcal{T}$ from the topos of sets to $\mathcal{T}$. To each point of $\mathcal{T}$ corresponds a contravariant functor $\mathcal{T} \longrightarrow \mathfrak{Sets}$ which is the inverse image part of the geometric morphism f and that preserves finite limits and arbitrary colimits. This picture generalizes the functor that associates to a sheaf of sets on a topological space X the stalk at a point of X. In particular, for the topos $\hat{\mathcal{C}} = \mathfrak{Sets}^{\mathcal{C}^{\mathrm{op}}}$ ($\mathcal{C}$ a small category), the points are described by *flat, covariant* functors $\mathfrak{F} : \mathcal{C} \longrightarrow \mathfrak{Sets}$. Then the inverse image part of the geometric morphism associated to a point of $\hat{\mathcal{C}}$ determines a natural generalization of the notion of the geometric realization of a simplicial set. This latter notion is obtained, in the case $\mathcal{C} = \Delta$, by considering the flat functor $\mathfrak{F} = \underline{\Delta} : \Delta \longrightarrow \mathfrak{Sets}$ that associates to an integer $n \geq 0$ the standard n-simplex. In general, the flatness of $\mathfrak{F}$ implies that the geometric realization functor

$$| \ | : \mathfrak{Sets}^{\mathcal{C}^{\mathrm{op}}} \longrightarrow \mathfrak{Sets}, \quad R \mapsto |R| := R \otimes_{\mathcal{C}} \mathfrak{F}$$

is left exact thus transforming finite products in $\mathfrak{Sets}^{\mathcal{C}^{\mathrm{op}}}$ into finite products in $\mathfrak{Sets}$. This property combines with the fact that $\mathfrak{F} = \underline{\Delta}$ extends to the larger cyclic category Λ to yield the natural action of the group $U(1)$ on the geometric realization of a cyclic set.

In this paper we provide a far reaching generalization of this construction for the points of the topos of simplicial sets. Given a point p of the topos $\hat{\Delta}$ with associated flat functor $\mathfrak{F} : \Delta \longrightarrow \mathfrak{Sets}$, a *cyclic structure* on p is defined to be an extension of $\mathfrak{F}$ from Δ to Λ. Our main result states that cyclic structures are classified by the datum provided by a totally left-ordered group G (not necessarily abelian) endowed with a central element $z > 1$ such that the interval $[1, z] \subset G$ generates G. It is well known (*cf.* [11]) that the points of the topos $\hat{\Delta} = \mathfrak{Sets}^{\Delta^{\mathrm{op}}}$ correspond to intervals I *i.e.* totally ordered sets with a minimal element b and a maximal element $t > b$. Proposition 4.2 shows that if (G, z) is a left-ordered group with a fixed central element $z > 1$, one obtains a natural cyclic structure on the point p_I of $\hat{\Delta}$ associated to the interval $I = [1, z]$. The converse of this statement constitutes the main result of this paper. More precisely one shows (*cf.* Theorem 5.1) the following

Theorem. *Let p be a point of the topos of simplicial sets and let I be the associated interval. Let $G = (\mathbb{Z} \times I)/\sim$ be endowed with the lexicographic ordering, where the equivalence relation identifies $(n, b) \sim (n-1, t)$, $\forall n \in \mathbb{Z}$. Then, a cyclic structure on p corresponds to a group law on G such that*

(*a*) *the order relation on G is left invariant*