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Completeness of the ring of polynomials

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ABSTRACT

Consider the polynomial ring $R := k[X_1, \ldots, X_n]$ in $n \geq 2$ variables over an uncountable field k. We prove that R is complete in its adic topology, that is, the translation invariant topology in which the non-zero ideals form a fundamental system of neighborhoods of 0. In addition we prove that the localization $R_\mathfrak{m}$ at a maximal ideal $\mathfrak{m} \subset R$ is adically complete. The first result settles an old conjecture of C.U. Jensen, the second a conjecture of L. Gruson. Our proofs are based on a result of Gruson stating (in two variables) that $R_\mathfrak{m}$ is adically complete when $R = k[X_1, X_2]$ and $\mathfrak{m} = (X_1, X_2)$.

1. Introduction

1. Consider for a field k and a given integer $n \geq 0$ the polynomial ring $R := k[X_1, \ldots, X_n]$ in n variables, and its field of fractions $K := k(X_1, \ldots, X_n)$. Set $d = 0$ if k is finite and define d by the cardinality equation $|k| = \aleph_d$ if k is infinite. The following conjecture in its full generality was formulated by L. Gruson (priv. com., 2013).

Conjecture. *In the notation above,* $\operatorname{Ext}^i_R(K, R) \neq 0 \iff i = \inf\{d + 1, n\}$.

The conjecture is trivially true for $n = 0$ where $R = K = k$ and the infimum equals 0. It is also true for $n = 1$ (where R is a PID and the infimum equals 1; the Ext may be computed from the injective resolution $0 \to R \to K \to K/R \to 0$).

In addition, the conjecture is trivially true if $i = 0$, since the infimum equals 0 iff $n = 0$.

The conjecture has an obvious analogue obtained by replacing the polynomial ring $R = k[X_1, \ldots, X_n]$ by its localization $R_\mathfrak{m}$ at a maximal ideal $\mathfrak{m}$.

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2. In this note we consider the conjectures only for $i = 1$. They were formulated some 40 years ago, Conjecture 2b partly by Gruson [1, p. 254], and Conjecture 2a by C.U. Jensen [2, p. 833], inspired by the work of Gruson.

Conjectures. *Let $R := k[X_1, \dots, X_n]$ be the polynomial ring, and $\mathfrak{m} \subset R$ a maximal ideal. Then the following bi-implications hold:*

2a. $\operatorname{Ext}^1_R(K, R) \neq 0 \iff n = 1$ *or* $|k| \leq \aleph_0$.
2b. $\operatorname{Ext}^1_{R_\mathfrak{m}}(K, R_\mathfrak{m}) \neq 0 \iff n = 1$ *or* $|k| \leq \aleph_0$.

Setup 3. The Ext's in the conjectures make sense for a wider class of rings, and we fix for the rest of this paper an integral domain R with field of fractions K. We assume that R is not a field; in particular, $\bigcap_{s \neq 0} sR = (0)$ and $\operatorname{Hom}_R(K, R) = 0$. Throughout we let $S := R \setminus \{0\}$ be the set of non-zero elements of R, pre-ordered by divisibility: $s' \mid s$ iff $sR \subseteq s'R$. We denote by $\varprojlim^{(i)}_S$ the i-th derived functor of the limit functor $\varprojlim_S$ on the category of inverse S-systems of R-modules.

The modules Ext^i of the conjectures are related to the modules $\varprojlim^{(i)}$ by well-known results, see [1, pp. 251–252]: For $i \geq 2$ there are natural isomorphisms $\operatorname{Ext}^i_R(K, R) \simeq \varprojlim^{(i-1)}_S R/sR$, and for $i = 1$ there is an exact sequence,

$$0 \to R \xrightarrow{c(R)} \varprojlim_{s \in S} R/sR \to \operatorname{Ext}^1_R(K, R) \to 0. \tag{3.1}$$

The set of principal ideals sR for $s \in S$ is cofinal in the set of all non-zero ideals of R. Hence the topology defined by the ideals sR for $s \in S$ is the *adic topology* on R, and the limit in (3.1) is the *adic completion* of R; we denote it by $\widehat{R}$, and we will simply call R *complete* if the *canonical injection* $c(R)$ in (3.1) is an isomorphism. As it follows from the exact sequence (3.1), R is complete iff $\operatorname{Ext}^1_R(K, R) = 0$.

Since R is not a field it follows easily that the completion $\widehat{R}$ is uncountable. Consequently, if the given field k is finite or countable (and $n \geq 1$) then the polynomial ring $R = k[X_1, \dots, X_n]$ and its localization $R_\mathfrak{m}$ are countable, and hence they are not complete. In other words, the assertions of Conjectures 2a and 2b hold if $|k| \leq \aleph_0$. As noted above, they also hold when $n \leq 1$. So the remaining cases of the conjectures are the following.

Conjectures. *Let $R := k[X_1, \dots, X_n]$ be the polynomial ring where $|k| \geq \aleph_1$ and $n \geq 2$. Then:*

3a. *(C.U. Jensen [2, Proposition 8, p. 833].) R is complete.*
3b. *The localization $R_\mathfrak{m}$ of R at any maximal ideal $\mathfrak{m} \subset R$ is complete.*

The main result of this paper is the verification of the two conjectures. In fact, both conjectures are implied by a single result.

Theorem 4. *Assume that $|k| \geq \aleph_1$, that $n \geq 2$, and that $R = U^{-1}R_0$ is a localization of $R_0 = k[X_1, \dots, X_n]$ with a multiplicative subset $U \subset R_0$. In addition, assume that every maximal ideal of R contracts to a maximal ideal of R_0. Then R is complete.*

The key ingredient in our proof is the following local result in two variables.

Proposition 5. *(See L. Gruson [1, Proposition 3.2, p. 252].) Conjecture* 3b *holds for $n = 2$ and $\mathfrak{m} = (X_1, X_2)$.*