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Invariants and conjugacy classes of triangular polynomial maps

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A B S T R A C T

In this article, we classify invariants and conjugacy classes of triangular polynomial maps. We make these classifications in dimension 2 over domains containing $\mathbb{Q}$, dimension 2 over fields of characteristic p, and dimension 3 over fields of characteristic zero. We discuss the generic characteristic 0 case. We determine the invariants and conjugacy classes of strictly triangular maps of maximal order in all dimensions over fields of characteristic p. These strictly triangular maps of maximal order turn out to be equivalent to a map of the form $(x_1 + f_1, \ldots, x_n + f_n)$ where $f_i \in x_n^{p-1} k[x_{i+1}^p, \ldots, x_n^p]$ if $1 \leq i \leq n-1$ and $f_n \in k^*$.

1. Introduction

1.1. Background

For notations and some definitions, please see Section 1.2 and further. Triangular polynomial maps are an important class of maps: they are the first non-affine, and in particular non-linear, polynomial automorphisms one comes up with, and they are a basic building block of many polynomial automorphisms. For one, in dimension two, all automorphisms are compositions of affine and triangular ones. Second, almost all basic examples are "almost triangular": for example, the Nagata automorphism is essentially a triangular automorphism conjugated in some way.

Due to polynomial automorphisms and endomorphisms in general being quite difficult, triangular polynomial maps are often considered trivial. For example, it's completely trivial to prove the Jacobian Conjecture for triangular polynomial endomorphisms. This is deceptive, however: if it is trivial to see that a polynomial endomorphism is an automorphism, doesn't necessarily make it easier to iterate it, or to find its invariants, or to find its conjugacy class. For all these last questions, there are some reasonably satisfactory answers one can give over fields of characteristic zero, or even rings or domains containing $\mathbb{Q}$ (see Section 2). Over fields of characteristic p this becomes much harder already. It's exactly this characteristic p case, especially the finite field case, which has gained more of an interest, also outside of the field of affine algebraic geometry [8,9,11].

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The overview of this paper is as follows:

In Section 1 we give background, introduction, definitions etc.

In Section 2 we elaborate on the characteristic 0 and rings-containing-$\mathbb{Q}$ case. If D is a locally nilpotent derivation, and $F = \exp(D)$, then we link the kernel and image of D to the image and kernel of $F - I$. We determine conjugacy classes in dimension 2 over general rings and in dimension 3 over fields.

Section 3 contains the most nontrivial result of this paper. We do in all dimensions the equivalent of the "locally nilpotent derivation having a slice"-case for characteristic p. Since there is no locally nilpotent derivation (or characteristic p equivalent of that), this case is truly different, and has a quite nontrivial proof and nontrivial answer. We provide a reasonable description of invariants, image and conjugacy classes for this case.

In Section 4 we determine the dimension 2 case over fields of characteristic p.

In Section 5 we briefly discuss automorphisms of finite order, and in Section 6 we give further research and acknowledgements.

1.2. Some notations and basic definitions

If R is a ring, we will denote $R[x_1, \ldots, x_n]$ as $R^{[n]}$. All rings in this article will be commutative with 1, and most of the time will be domains. We will reserve k for a field. We define $\mathrm{GA}_n(R)$ as the set of polynomial automorphisms of $R^{[n]}$, and elements $F \in \mathrm{GA}_n(R)$ as $F = (F_1, \ldots, F_n)$ where $F_i \in k^{[n]}$. $\mathrm{BA}_n(R)$ is the set of triangular polynomial automorphisms, i.e. where $F_i \in k[x_i, x_{i+1}, \ldots, x_n]$. (BA stands for Borel Automorphisms, see [1].) It follows that $F_i = a_i x_i + f_i$ where $f_i \in k[x_{i+1}, \ldots, x_n]$. The group $\mathrm{BAs}_n(R)$ is the set of strictly upper triangular polynomial maps, i.e. maps of the form $F = (x_1 + f_1, \ldots, x_n + f_n)$ where $f_i \in k[x_{i+1}, \ldots, x_n]$. $\mathrm{Aff}_n(R)$ is the set of affine maps, i.e. compositions of linear maps and translations.

Whenever we write F^i for some polynomial map F and integer i, then this means iterative composition (i times).

1.3. Unipotent and triangular maps

Definition 1.1. Let $F \in \mathrm{GA}_n(R)$. Then F is called *locally finite* (short LF) if there exist $d \in \mathbb{N}$ and $a_i \in R$ such that $F^d = \sum_{i=0}^{d-1} a_i F^i$. It follows that $\deg(F^m)$ is bounded. In case the polynomial $T^d - \sum_{i=0}^{d-1} a_i T^i = (T-1)^d$, then we say that F is unipotent.

(Note: in some articles, LF is called "algebraic", see for example [6].)

Example 1.2. All elements in $\mathrm{BA}_n(R)$ are locally finite. The elements in $\mathrm{BAs}_n(R)$ are unipotent.

It will be convenient to abbreviate the notation for the elements in $\mathrm{BAs}_n(k)$ which have many identity components, for example

$$(x_1, \ldots, x_{i-1}, x_i + f_i, x_{i+1}, \ldots, x_n) = (x_i + f_i)$$

$$(x_1, \ldots, x_{i-1}, x_i + f_i, x_{i+1}, \ldots, x_{j-1}, x_j + f_j, x_{j+1}, \ldots, x_n) = (x_i + f_i, x_j + f_j)$$

etc.

If $A \subseteq R^{[n]}$ then we denote $A^F = \{a \in A \mid F(a) = a\}$. In case A is clear (mostly, meaning $A = R^{[n]}$) then we write $\mathrm{inv}(F) = A^F$.

In this article, our goal is to understand elements in $\mathrm{BAs}_n(k)$ for any field k. In particular, we want to understand the following: