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Equality of linear and symplectic orbits

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It is shown that the set of orbits of the action of the elementary symplectic transvection group on all unimodular elements of a symplectic module over a commutative ring in which 2 is invertible is identical with the set of orbits of the action of the corresponding elementary transvection group. This result is used to get improved injective stability estimates for K_1 of the symplectic transvection group over a non-singular affine algebras.

1. Introduction

In this paper we discuss two related questions about (linear and symplectic) elementary transvections of a projective R-module (resp. symplectic R-module) of type $R \oplus P$ (resp. $(\mathbb{H}(R) \oplus (P, \langle , \rangle))$).

The first one is about comparing the groups generated by the two different types of elementary transvections; and showing that they are the same. (This fact seems to have escaped notice earlier; and experts have told us that it is interesting, and opens up the study done on these transvections.)

The second one is to show that the linear and symplectic elementary transvection orbits of a unimodular element in a symplectic module coincide. (This generalizes the result in [7] where it was shown in the free case.)

We now describe the two problems a bit more in detail.

H. Bass introduced two types of linear transvections of a projective module $R \oplus P$ in [3]. He also introduced two types of symplectic transvections of a symplectic module $\mathbb{H}(R) \oplus (P, \langle , \rangle)$. (These are recalled in Section 4 and Section 5.)

Since elementary automorphisms are homotopic to the identity, we are able to invoke Quillen–Suslin theory (see [11,13]) to show that

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- the groups generated by the two types of elementary linear transvections are the same as the elementary linear group in the free case (see Lemma 4.5);
- the group generated by the two types of elementary symplectic transvections w.r.t. the standard alternating form are the same as the elementary symplectic group in the free case (see Lemma 5.14).

The above generalizes the special case of these results in ([6], Theorem 2).

The title of this paper alludes to the comparison of the elementary linear and elementary symplectic orbits of a unimodular element (a, b, p) in a symplectic module $(\mathbb{H}(R) \oplus P)$. In case P is free of rank ≥ 4, it is established in ([7], Theorem 4.2, Theorem 5.6) that these two orbits coincide. In Appendix A the missing case when P is free of rank 2 is proved by a similar, but slightly more involved argument. (This means one has to essentially prove Lemma 2.9 and Lemma 3.1 in [7].)

Since elementary automorphisms are homotopic to the identity, we show how the Quillen–Suslin machinery enables one to extend the results of [7] to show that the two orbits are equal in the general case when P is a finitely generated projective module. (The transition is by no means automatic!)

2. Preliminaries

A row $v = (v_1, \dots, v_n) \in R^n$ is said to be *unimodular* if there are elements $w_1, \dots, w_n$ in R such that $v_1 w_1 + \cdots + v_n w_n = 1$. $\mathrm{Um}_n(R)$ will denote the set of all unimodular rows $v \in R^n$. Let I be an ideal in R. We denote by $\mathrm{Um}_n(R, I)$ the set of all unimodular rows of length n which are congruent to $e_1 = (1, 0, \dots, 0)$ modulo I. (If $I = R$, then $\mathrm{Um}_n(R, I)$ is $\mathrm{Um}_n(R)$.)

Definition 2.1. Let P be a finitely generated projective R-module. An element $p \in P$ is said to be *unimodular* if there exists an R-linear map $\phi : P \to R$ such that $\phi(p) = 1$. The collection of unimodular elements of P is denoted by $\mathrm{Um}(P)$.

Let P be of the form $R \oplus Q$ and have an element of the form $(1, 0)$ which correspond to the unimodular element. An element $(a, q) \in P$ is said to be *relative unimodular* w.r.t. an ideal I of R if (a, q) is unimodular and (a, q) is congruent to $(1, 0)$ modulo IP. The collection of all relative unimodular elements w.r.t. an ideal I is denoted by $\mathrm{Um}(P, IP)$.

Let us recall that if M is a finitely presented R-module and S is a multiplicative set of R, then $S^{-1}\mathrm{Hom}_R(M, R) \cong \mathrm{Hom}_{R_S}(M_S, R_S)$. Also recall that if $f = (f_1, \dots, f_n) \in R^n := M$, then $\Theta_M(f) = \{\phi(f) : \phi \in \mathrm{Hom}(M, R)\} = \sum_{i=1}^{n} Rf_i$. Therefore, if P is a finitely generated projective R-module of rank n, $\mathfrak{m}$ is a maximal ideal of R and $v \in \mathrm{Um}(P)$, then $v_{\mathfrak{m}} \in \mathrm{Um}_n(R_{\mathfrak{m}})$. Similarly if $v \in \mathrm{Um}(P, IP)$ then $v_{\mathfrak{m}} \in \mathrm{Um}_n(R_{\mathfrak{m}}, I_{\mathfrak{m}})$.

The group $\mathrm{GL}_n(R)$ of invertible matrices acts on R^n in a natural way: $v \longrightarrow v\sigma$, if $v \in R^n$, $\sigma \in \mathrm{GL}_n(R)$. This map preserves $\mathrm{Um}_n(R)$, so $\mathrm{GL}_n(R)$ acts on $\mathrm{Um}_n(R)$. Note that any subgroup G of $\mathrm{GL}_n(R)$ also acts on $\mathrm{Um}_n(R)$. Let $v, w \in \mathrm{Um}_n(R)$, we denote $v \sim_{\mathrm{G}} w$ or $v \in w\mathrm{G}$ if there is a $g \in \mathrm{G}$ such that $v = wg$.

Let $\mathrm{E}_n(R)$ denote the subgroup of $\mathrm{SL}_n(R)$ consisting of all *elementary* matrices, i.e. those matrices which are a finite product of the *elementary generators* $\mathrm{E}_{ij}(\lambda) = I_n + e_{ij}(\lambda)$, $1 \leq i \neq j \leq n$, $\lambda \in R$, where $e_{ij}(\lambda) \in \mathrm{M}_n(R)$ has an entry λ in its (i, j)-th position and zeros elsewhere.

In the sequel, if α denotes an $m \times n$ matrix, then we let α^t denote its *transpose* matrix. This is of course an $n \times m$ matrix. However, we will mostly be working with square matrices, or rows and columns.

Definition 2.2 *(The relative groups* $\mathbf{E}_n(I)$, $\mathbf{E}_n(R, I)$*).* Let I be an ideal of R. The relative elementary group $\mathrm{E}_n(I)$ is the subgroup of $\mathrm{E}_n(R)$ generated as a group by the elements $\mathrm{E}_{ij}(x)$, $x \in I$, $1 \leq i \neq j \leq n$.

The relative elementary group $\mathrm{E}_n(R, I)$ is the normal closure of $\mathrm{E}_n(I)$ in $\mathrm{E}_n(R)$.