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Closed polynomials in polynomial rings over integral domains ☆

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ABSTRACT

Let f be a non-constant polynomial in the polynomial ring $R^{[n]}$ in n variables over an integral domain R such that $Q(R)[f] \cap R^{[n]} = R[f]$. We give necessary and sufficient conditions for f to be closed in $R^{[n]}$.

1. Introduction

Let R be an integral domain with unit. We denote by $R^{[n]}$ the polynomial ring in n variables over R and by $Q(R)$ the field of fractions of R. A non-constant polynomial $f \in R^{[n]} \setminus R$ is said to be *closed in* $R^{[n]}$ if the ring $R[f]$ is integrally closed in $R^{[n]}$.

When R is a field, closed polynomials in $R^{[n]}$ have been studied by several mathematicians. See, e.g., Nowicki [10], Nowicki–Nagata [12], Ayad [2], Arzhantsev–Petravchuk [1], Jędrzejewicz [6], etc. By virtue of these papers, we have several characterizations of closed polynomials. See, e.g., [1, Theorem 1] for more details. In [7], Kato and the first author studied closed polynomials in $R^{[n]}$ when R is a UFD and generalized some results of [12] and [1].

In this paper, we study closed polynomials in $A := R^{[n]}$ for any integral domain R and generalize some results of [7]. In Section 2, integrally closed R-subalgebras of A are studied. We give in Theorem 2.1 a necessary and sufficient condition for an R-subalgebra of A to be expressed as the kernel of some rational higher R-derivation on A satisfying a condition. By virtue of [8, Theorem 1.1], Theorem 2.1 includes [8, Theorem 1.2]. In the case where the characteristic of R equals zero, we give in Theorem 2.2 a necessary

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and sufficient condition for an R-subalgebra of A to be expressed as the kernel of some R-derivation on A. In Section 3, closed polynomials in A are studied. For a non-constant polynomial $f \in A \setminus R$ such that $Q(R)[f] \cap A = R[f]$, we give in Theorem 3.1 necessary and sufficient conditions for the polynomial f to be closed in A. Then, by using this result, we generalize some results of [10, Section 2]. In Section 4, we give some remarks on the hypotheses of Theorem 3.1.

2. Integrally closed subalgebras of polynomial rings

First of all, we recall some notions on higher derivations. For more details, we refer to [8] and [9].

Let R be an integral domain with unit and A an R-domain. Let F be an integral domain that contains A as an R-subalgebra. A set $D = \{D_\ell\}_{\ell=0}^{\infty}$ of R-homomorphisms from A to F is said to be a *higher R-derivation on A into F* if the following conditions are satisfied:

(1) D_0 is the identity map of A;
(2) For any $a, b \in A$ and for any integer $\ell \geq 0$, we have

$$D_\ell(ab) = \sum_{i+j=\ell} D_i(a)D_j(b).$$

A higher R-derivation on A into F is called a *higher R-derivation on A* (resp. a *rational higher R-derivation on A*) if $F = A$ (resp. $F = Q(A)$). For a higher R-derivation $D = \{D_\ell\}_{\ell=0}^{\infty}$ on A into F, we define the kernel A^D of D by $\{a \in A \mid D_\ell(a) = 0 \text{ for any } \ell \geq 1\} = \cap_{\ell \geq 1}\mathrm{Ker}\, D_\ell$.

Let $D = \{D_\ell\}_{\ell=0}^{\infty}$ be a rational higher R-derivation on A and let $\varphi_D : A \to Q(A)[[t]]$, where $Q(A)[[t]]$ is the formal power series ring in one variable over $Q(A)$, be the mapping defined by $\varphi_D(a) = \sum_{i\geq 0} D_i(a)t^i$ for $a \in A$. Since D is a rational higher R-derivation on A, φ_D is a homomorphism of R-algebras. We call the mapping φ_D the *homomorphism associated to D*.

For a rational higher R-derivation $D = \{D_\ell\}_{\ell=0}^{\infty}$ on A, we have a unique higher $Q(R)$-derivation $\overline{D} = \{\overline{D}_\ell\}_{\ell=0}^{\infty}$ on $Q(A)$ such that $\overline{D}_\ell|_A = D_\ell$ for any $\ell \geq 0$. We call $\overline{D}$ the *extension of D to $Q(A)$*. For more details on the construction of $\overline{D}$, we refer to [8, Section 1]. It is clear that $Q(A^D) \subset Q(A)^{\overline{D}}$.

We note that all the results of [9] remain true if we assume that D (with the notation of [9]) is a rational higher R-derivation. So we use the results of [9] assuming that D is a rational higher R-derivation.

We prove the following result.

Theorem 2.1. *Let R be an integral domain and $A = R^{[n]}$ the polynomial ring in n variables over R. Let B be an R-subalgebra of A. Then the following conditions are equivalent to each other.*

(1) *B is integrally closed in A, $Q(B) \cap A = B$ and $Q(A)/Q(B)$ is a separable field extension.*
(2) *There exists a rational higher R-derivation D on A such that $B = A^D$ and $Q(A)^{\overline{D}} = Q(B)$, where $\overline{D}$ denotes the extension of D to $Q(A)$.*

Proof. Set $K = Q(R)$, $B_K = K \otimes_R B$ and $A_K = K \otimes_R A = K^{[n]}$. Then B_K is a K-subalgebra of A_K.

(2) $\Longrightarrow$ (1) By [9, Lemma 2.2 (1)], B is integrally closed in A. By [9, Lemma 2.3 (2)], $Q(A)^{\overline{D}} \cap A = Q(A^D) \cap A = A^D = B$. Since $Q(B) = Q(A)^{\overline{D}}$ and $\overline{D}$ is a higher K-derivation on $Q(A)$, it follows from [5, Theorem (2.3)] that $Q(A)/Q(B)$ is a regular field extension. In particular, the field extension $Q(A)/Q(B)$ is separable.

(1) $\Longrightarrow$ (2) Since A_K is normal and B is integrally closed in A, we know that $Q(B)$ is algebraically closed in $Q(A)$. So $Q(A)/Q(B)$ is a regular field extension. It then follows from [14, Theorem 1] that there exists a higher K-derivation $\tilde{D} = \{\tilde{D}_\ell\}_{\ell=0}^{\infty}$ on $Q(A)$ such that $Q(A)^{\tilde{D}} = Q(B)$. Set $D_\ell = \tilde{D}_\ell|_A$ for each