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Orbital characters and their applications ☆

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A B S T R A C T

In this paper, we provide some formulae to calculate orbital characters. Especially we give a lifting theorem by which one can read off the value of orbital characters from local subgroups. With orbital characters, we can show that Reynolds ideal is dual to the vector space of simple modules and this leads to a new method to count simple modules. We also give a natural morphism from local to general vector space of simple modules and determine it completely. Some other applications to modular representation are also included.

1. Introduction

Let (K, R, F) be a modular system. The characteristic of K is zero and that of F is p (a prime). Let G be a finite group. Assume that the modular system is large enough for G. We denote R or F by $\mathcal{O}$.

Orbital characters of permutation modules were first introduced by Scott in [14], they were used to study the primitive groups of degree kp. Roughly speaking, an orbital character is a symmetric linear function on $\mathcal{O}G$ which is defined by an element in the endomorphism ring of a permutation $\mathcal{O}G$-module. Similarly we can define orbital characters for any $\mathcal{O}G$-modules. In fact a character afforded by $\mathcal{O}G$-module M is just the orbital character associated with the identity morphism of M.

In this paper, we will give some properties of orbital characters and formulae to calculate orbital characters afforded by permutation modules. In particular, we give a lifting theorem for orbital characters by which one can read off the value of orbital character from local subgroups and we can lift local properties to the group G.

With orbital characters we provide something new in modular representation theory. At first we show that the Reynolds ideal is dual to the vector space of simple modules as ZFG-modules. Then we define a natural map Z_Q from simple module vector space of local subgroups to the dual space of simple module

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vector space of G and give an explicit description. It can be seen as a connection between simple modules of local subgroups and simple modules of G.

To count the p-blocks with given defect group and the irreducible characters in blocks are two important problems in modular representation. There are many papers on these problems such as [3,7,11,13,12,16,17]. Based on orbital characters we can put these problems into more natural and uniform settings and give more subtle countings. The number of irreducible characters is characterized by the rank of a matrix $M(D)$. With notion in [12], we can simplify $M(D)$ to a matrix A_D which is the same as matrix in [12] up to a transpose. In fact this method can be used to count the number of simple modules in p-blocks of any finite dimensional algebra over a field. Given an FG-module M, we can give a linear map

$$\tau_M : End_F(M) \to Z(FG),$$

which is expressed by orbital characters. Since orbital characters over F always vanish on the radical of $End_{FG}(M)$, τ_M has good properties. We can prove that the matrix of τ_M is just Robinson's matrix defined in [11] whose rank is the number of blocks with given defect group. This gives a natural meaning of Robinson's matrix. We always restrict us to local subgroups in case of counting p-blocks. With the lift theorem of orbital characters, we can get rid of this restriction and lift Robinson's matrix from local case to general setting.

Let $N_{H,D}$ be Robinson's matrix (see the definition in Section 7). Then the p-rank of $N_{H,D}N'_{H,D}$ is the number of blocks with D as defect group in case that $H \in Syl_p(G)$ and $D \lhd G$. In our proof of Robinson's theorem, we find that if H is not a Sylow p-subgroup, the p-rank of $N_{H,D}N'_{H,D}$ is zero. So we cannot determine the number of blocks with Robinson's matrix in the case H is not Sylow p-subgroup but H contains the defect group D. Since the projective simple modules are direct summands of permutation modules induced from any non-trivial p-subgroup Q, we should have a method to determine the number of blocks of defect zero from the double cosets of Q. More generally, we have the following problem.

Problem. How to describe the number of blocks with defect group D by using the double cosets of H, where H is p-subgroup containing D but H is not Sylow p-subgroup?

We will give an answer for this problem in this paper. A matrix counting the weights in local subgroups is also constructed.

Here is the outline of the sections. In Section 2, we focus on basic properties and calculations of orbital characters. In Section 3, we gives some notations on double cosets and collect some results on S_A, T_A. The main purpose of this section is to show the Reynolds ideal is dual to the vector space of simple modules. In Section 4, we define a natural morphism from local to general vector spaces of simple modules and determine the exact image of this morphism. In Section 5, by using dual property we give a method to determine the number of simple modules in some block ideals. A matrix similar to that in [13] is constructed to counting the number of simple modules in p-blocks. In Section 6, we study the relationship between orbital characters of G and orbital characters of its local subgroups. A lifting formula on this relationship is proven. In Section 7, a new proof of Robinson's theorem on number of blocks with given defect group is provided by calculating orbital characters and we use formula in Section 6 to lift it to general case. In Section 8, the problem put forward above is solved by defining a matrix of double cosets of H to count the number of blocks. In Section 9, we give a matrix to count the number of weights at local level.

Concerning notations and terminology, we refer to [4,8,10,14].

2. Orbital character

Assume that G acts on the finite set Ω transitively. The action of G on Ω gives rise to a natural algebra homomorphism