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An effective bound for reflexive sheaves on canonically trivial 3-folds

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ABSTRACT

We give effective bounds for the third Chern class of a semistable rank 2 reflexive sheaf on a canonically trivial threefold.

1. Introduction

We work over an algebraically closed field of characteristic 0.

In a previous work [5] we showed that if $\mathcal{F}$ is a semistable rank 2 reflexive sheaf on a smooth projective threefold X with $\operatorname{Pic}(X) = \mathbb{Z}$, then there is an upper bound on $c_3(\mathcal{F})$ in terms of $c_i(X)$, $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$ (note that $c_3(\mathcal{F}) \geqslant 0$). It has more recently been conjectured [2] that this remains true for any smooth, projective threefold. Here we derive explicit effective bounds in the case of a polarized smooth projective threefold X with $\omega_X = \mathcal{O}_X$, without the restriction on the Picard group.

2. Stability and boundedness

Definition 1. Let L be a very ample line bundle on a smooth projective variety X. A rank 2 reflexive coherent sheaf $\mathcal{F}$ on X is ***L*-stable** (resp. ***L*-semistable**) if for every invertible subsheaf $\mathcal{F}'$ of $\mathcal{F}$ with $0 < \operatorname{rank} \mathcal{F}' < \operatorname{rank} \mathcal{F}$, we have $\mu(\mathcal{F}', L) < \mu(\mathcal{F}, L)$ (resp. $\leqslant$), where

$$\mu(\mathcal{F}, L) = \frac{c_1(\mathcal{F}).[L]^{\dim X - 1}}{(\operatorname{rank} \mathcal{F})[L]^{\dim X}} \qquad \square$$

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Definition 2. A reflexive sheaf $\mathcal{F}$ is **normalized with respect to L** if $-1 < \mu(\mathcal{F}, L) \leqslant 0$. As L is typically fixed, we usually say simply that $\mathcal{F}$ is **normalized**. Note that as $\mu(\mathcal{F} \otimes L, L) = \mu(\mathcal{F}, L) + 1$, there exists, for any fixed $\mathcal{F}$, a unique $k \in \mathbb{Z}$ such that $\mathcal{F} \otimes L^k$ is normalized with respect to L. □

For a fixed smooth, canonically trivial $X \subset \mathbb{P}^n$, our goal is to give a bound on $c_3(\mathcal{F})$ in terms of $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$.

Lemma 3. *Let X be a smooth projective canonically trivial threefold, L any very ample line bundle on X, $\mathcal{F}$ a rank two reflexive sheaf. If $\mathcal{F}$ admits a section $s \in \Gamma(\mathcal{F})$ whose zero locus is a curve Y, then*

$$c_3(\mathcal{F}) \leqslant d^2 - 3d - c_1(\mathcal{F})c_2(\mathcal{F})$$

where $d = c_1(L)c_2(\mathcal{F})$ is the degree of Y in the embedding induced by L.

Proof. This is directly related to the Hartshorne–Serre correspondence [3, Theorem 4.1]. It is shown there that any such Y is locally Cohen–Macaulay, and it can be immediately deduced [5, Theorem 1] that $c_3(\mathcal{F}) = 2p_a(Y) - 2 - c_2(\mathcal{F})c_1(\omega_X) - c_1(\mathcal{F})c_2(\mathcal{F})$. In the canonically trivial case $c_2(\mathcal{F})c_1(\omega_X) = 0$, and in any case the degree of the curve section Y in the embedding given by L is $d = c_1(L)c_2(\mathcal{F})$. The fact that $2p_a(Y) - 2 \leqslant d^2 - 3d$ is the bound coming from the degree of a plane curve. □

In some common situations, we can do better:

Proposition 4. *Let X be a smooth projective canonically trivial threefold, L a very ample line bundle, $\mathcal{F}$ a stable (resp. semistable) rank two reflexive sheaf. Suppose that $\mu(\mathcal{F}, L) \geqslant 1$ (resp. $\mu(\mathcal{F}, L) > 1$) and that $H^1(X, \det \mathcal{F}^* \otimes L) = 0$. If $s \in \Gamma(\mathcal{F})$ is a section whose zero locus is a smooth curve Y, then*

$$c_3(\mathcal{F}) \leqslant m(m-1)\big(h^0(X, L) - 2\big) + 2m\epsilon - 2 - c_1(\mathcal{F})c_2(\mathcal{F})$$

where

$$m = \left\lfloor \frac{c_1(L)c_2(\mathcal{F}) - 1}{h^0(X, L) - 2} \right\rfloor$$
$$\epsilon = c_1(L)c_2(\mathcal{F}) - 1 - m\big(h^0(X, L) - 2\big)$$

Proof. The section gives the sequence

$$0 \to \det \mathcal{F}^* \otimes L \to \mathcal{F}^* \otimes L \to \mathcal{I}_Y \otimes L \to 0$$

By hypothesis, $\mu(\mathcal{F}^* \otimes L, L) \leqslant 0$ when $\mathcal{F}$ is stable and $\mu(\mathcal{F}^* \otimes L, L) < 0$ when $\mathcal{F}$ is semistable. In either case $H^0(X, \mathcal{F}^* \otimes L) = 0$, and so $H^0(X, \mathcal{I}_Y \otimes L) = 0$. This implies that Y is a non-degenerate curve in the embedding induced by L, hence we may apply Castelnuovo's Bound [1, p. 116]. □

The idea now is: given a very ample line bundle L, bound the twist of $\mathcal{F}$ by L^r needed to produce a section, and then use the bound in Lemma 3. We do this by first finding a bound for the vanishing of $h^2(\mathcal{F} \otimes L^r)$ and then by making the Euler characteristic positive.

The next two results (Proposition 5 and Corollary 7) follow directly from more general results in [5]. The first proceeds by showing that a minimal m satisfying the given inequality but contradicting the claim must be positive; the second proceeds from the first by Serre Duality on the smooth surface D, and then the stated inequality forces the Euler characteristic to be positive via Riemann–Roch, hence the relevant h^0 must be strictly greater than h^1, and in particular must be positive.