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On the fundamental theorem of algebra for polynomial equations over real composition algebras

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A R T I C L E I N F O

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A B S T R A C T

A topological proof is given that real compositions algebras of finite dimension greater than one are algebraically closed under polynomial equations with a tame tail.

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1. Introduction

In this paper, we are concerned with the problem of existence of solutions of certain polynomial equations over a class of real division algebras called composition algebras. The most important result regarding composition algebras themselves is due to Hurwitz [10] who proved that every finite-dimensional real composition algebra with identity is isomorphic to one of the four algebras: the reals $\mathbb{R}$, the complexes $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. The historical origins of the term composition algebra have roots in Gauss's work on composition theory of integral quadratic forms which was published in 1801 as a section in his famous *Disquisitiones arithmeticae*.

The fundamental theorem of algebra, asserting that every nonconstant complex polynomial has at least one zero in $\mathbb{C}$, also goes back to Gauss who gave its first formal proof [6]. Before Gauss published his proof, many well-known mathematicians such as Euler, d'Alembert, Lagrange and Laplace had attempted their own proofs, but most of those attempts were only partially successful. Literally hundreds of various proofs have been published since Gauss's original proof, including three more proofs by Gauss himself. The interested reader may find more information about the history of this important result and the interesting personalities involved by consulting Refs. [3] and [1].

Polynomial equations in noncommutative division rings on the other hand have not received much attention until Niven's paper [13]. Eilenberg and Niven published a proof of the fundamental theorem of algebra for quaternions in [4]. Their result stated that every nonconstant quaternionic polynomial, with the highest degree term a monomial, has at least one zero in $\mathbb{H}$. The restriction on the type of the polynomial was significant there since certain polynomial equations over $\mathbb{H}$ were known not to have any solutions in $\mathbb{H}$ nor in any ring extension of $\mathbb{H}$. An extension of this result to octonions was published by Jou in [11] under a similar restriction. Some other refinements of [4] can be found in [5] and [15], with further additions and generalizations to other division rings or algebras in [7,9,16]. Ref. [12] is a survey article about polynomial equations in division rings. Our main result is that real composition algebras of finite dimension greater than one are algebraically closed under polynomial equations with a tame tail.

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Theorem 1.1. *If A is a real composition algebra of finite dimension greater than one, then every polynomial function $f : A \to A$ with a tame tail is surjective and in particular has at least one zero.*

All technical terms in this statement will be explained later in the paper. We only note here that polynomial functions with a tame tail include all nonconstant complex polynomials as well as all nonconstant quaternionic and octonionic polynomials with the highest degree term a monomial. Several examples will be given to demonstrate that this is in fact a much larger class of polynomial functions. Thus our result generalizes [4] and [11] by allowing a more general domain of coefficients and extends those results at the same time to a larger class of equations.

Part of the difficulty in dealing with polynomial equations over a noncommutative algebra is that the traditional (i.e., commutative) polynomials do not generate enough polynomial functions. That is, the collection of polynomial functions so obtained is not closed under pointwise function multiplication and thus is not an algebra. The most obvious remedy for this deficiency would be to introduce more polynomials. A construction due to Gordon and Motzkin [7] accomplished this in the case of an associative algebra with identity. However, many composition algebras are nonassociative and/or lack a unit so the results of [7] are not applicable in such cases. To deal with the general case, we introduce the concept of a formal polynomial algebra whose suitable quotient is the required polynomial function algebra. A related but different construction of this type can be also found in [16].

2. Free algebras and coproducts

Let K be a commutative ring with identity. In this paper, a K-*algebra* is a unitary (left) K-module A equipped with a K-bilinear multiplication map $A \times A \to A$, $(a,b) \mapsto ab$. It is not assumed that this multiplication is commutative or associative or that A possess a multiplicative unit. An algebra A which has a two-sided multiplicative identity element will be called a *unitary* algebra. A K-module homomorphism $\phi : A \to B$ between K-algebras is a K-algebra homomorphism provided that $\phi(ab) = \phi(a)\phi(b)$ holds for all $a, b \in A$. K-Algebras and their homomorphisms form a category which will be denoted by $\mathcal{A}(K)$. This category has arbitrary products, coproducts as well as free objects. These universal objects in $\mathcal{A}(K)$ can be uniquely characterized up to isomorphism by their corresponding universal properties, once their existence has been established (see [14]). Some applications and calculations we wish to carry out require a concrete model of the coproduct, so we include here a brief discussion of one such construction.

We begin by introducing a bit of useful notation. Let Γ_m denote the finite set of all formal groupings of an ordered product of m elements that can be inductively formed in an arbitrary binary algebra. For example, Γ_3 consists of two elements corresponding to the two possible products $(ab)c$ and $a(bc)$ of length 3. For $m < 3$, Γ_m has of course only one element. If $(a_1, \dots, a_m)$ is an ordered m-tuple of elements of an algebra A and $\gamma \in \Gamma_m$, let $(a_1, \dots, a_m)^\gamma$ denote the unique element of A obtained by multiplying those elements in the given order with grouping γ. Groupings can be composed in a natural way: if $\gamma \in \Gamma_m$ and $\gamma' \in \Gamma_n$, then $\gamma * \gamma'$ determines a grouping of a product of $m+n$ elements according to the formula

$$\big((a_1, \dots, a_m)^\gamma\big)\big((a'_1, \dots, a'_n)^{\gamma'}\big) = \big(a_1, \dots, a_m, a'_1, \dots, a'_n\big)^{\gamma * \gamma'}.$$

This composition defines a function $\Gamma_m \times \Gamma_n \to \Gamma_{m+n}$ which is neither commutative nor associative.

Given a nonempty set X, let $K\langle((\mathrm{X}))\rangle$ denote the free K-module generated by all $(m+1)$-tuples of the form $(\mathrm{x}_1, \dots, \mathrm{x}_m, \gamma)$, where $\mathrm{x}_i \in \mathrm{X}$, $\gamma \in \Gamma_m$ and $m \geqslant 1$. We can view X as a subset of $K\langle((\mathrm{X}))\rangle$ by identifying each $\mathrm{x} \in \mathrm{X}$ with (x, γ). The formula

$$(\mathrm{x}_1, \dots, \mathrm{x}_m, \gamma) \cdot \big(\mathrm{x}'_1, \dots, \mathrm{x}'_n, \gamma'\big) = \big(\mathrm{x}_1, \dots, \mathrm{x}_m, \mathrm{x}'_1, \dots, \mathrm{x}'_n, \gamma * \gamma'\big)$$

extends to a K-algebra multiplication in $K\langle((\mathrm{X}))\rangle$. It follows by induction that

$$(\mathrm{x}_1, \dots, \mathrm{x}_m, \gamma) = (\mathrm{x}_1, \dots, \mathrm{x}_m)^\gamma$$

holds for all $\gamma \in \Gamma_m$. Thus each element of $K\langle((\mathrm{X}))\rangle$ can be uniquely written as a finite sum $\sum_{I,\gamma} k_{I,\gamma} (\mathrm{x}_{i_1}, \dots, \mathrm{x}_{i_m})^\gamma$, where $k_{I,\gamma} \in K$, $I = (i_1, \dots, i_m)$ and $\gamma \in \Gamma_m$.

Proposition 2.1. *$K\langle((\mathrm{X}))\rangle$ is a free K-algebra on X. That is, every function $\psi : \mathrm{X} \to A$ into a K-algebra extends to a unique K-algebra homomorphism $\psi_* : K\langle((\mathrm{X}))\rangle \to A$.*

Proof. Let $\psi_*(\sum_{I,\gamma} k_{I,\gamma} (\mathrm{x}_{i_1}, \dots, \mathrm{x}_{i_m})^\gamma) = \sum_{I,\gamma} k_{I,\gamma} (\psi(\mathrm{x}_{i_1}), \dots, \psi(\mathrm{x}_{i_m}))^\gamma$. □

Proposition 2.2. *Every indexed family of K-algebras $\{A_\lambda\}_{\lambda \in \Lambda}$ has a coproduct in $\mathcal{A}(K)$. That is, there exists a K-algebra $A = \coprod_{\lambda \in \Lambda} A_\lambda$, together with K-algebra homomorphisms $\{\iota_\lambda : A_\lambda \to A\}_{\lambda \in \Lambda}$ such that given a family of K-algebra homomorphisms $\{\phi_\lambda : A_\lambda \to B\}_{\lambda \in \Lambda}$, there is a unique K-algebra homomorphism $\phi : A \to B$ with $\phi \circ \iota_\lambda = \phi_\lambda$ for all $\lambda \in \Lambda$.*

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