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Some stable homology calculations and Occam's razor for Hodge structures

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ABSTRACT

Motivated by motivic zeta function calculations, Vakil and Wood in [10] made several conjectures regarding the topology of subspaces of symmetric products. The purpose of this note is to prove two of these conjectures and disprove a strengthening of one of them.

1. Introduction

In [10] Vakil and Wood introduced a method for predicting the rational homology groups of subspaces of symmetric products of complex varieties and used it to formulate several conjectures regarding these homology groups. This method uses motivic zeta functions and a principle they dub "Occam's razor for Hodge structures." Roughly, the idea of this principle is as follows. For some finite type varieties X over a field one can compute the Hodge–Deligne polynomial, which is given by the two-variable polynomial $\sum_k (-1)^k h^{p,q}(Gr_W^{p+q} H_c^*(X,\mathbb{Q}))\, x^p y^q$. That is, it is the generating function with coefficient of $x^p y^q$ equal to the dimension of p-part of the Hodge filtration in the $(p+q)$th associated graded of the weight filtration. If X were smooth and proper these are the classical Hodge numbers, but for varieties that are not smooth and proper one cannot determine the Betti numbers from the Hodge–Deligne polynomial without more information about the Hodge structure. In many cases there is however a simplest choice of Hodge structure compatible with the Hodge–Deligne polynomial and Occam's razor for Hodge structures obtains predictions by assuming that this simplest choice is the correct one. For certain families of locally closed subvarieties of symmetric products, Vakil and Wood are able to determine the limiting Hodge–Deligne polynomial using the theory of motivic zeta functions. Using this and Occam's razor for Hodge structures, they are able to make conjectures about the limiting homology of these families of subvarieties.

We prove two of Vakil and Wood's conjectures, namely Conjecture G and Conjecture H. This can be seen as giving evidence supporting the predictive power of Occam's razor for Hodge structures. On the other hand, we also show that Formula 1.50 of [10] – here Eq. (1) – is incorrect, giving an example where Occam's razor for Hodge structures predicts the wrong answer.

We now give a definition of the types of subspaces of symmetric products to which the conjectures pertain. Recall that the kth fold symmetric product of a space X, denoted $Sym^k(X)$, is the quotient of X^k by the natural action of the symmetric group on k letters. A point $\xi = (x_1, \dots, x_k) \in Sym^k X$ determines a partition of the number k by recording the multiplicity of each point x_i appearing in ξ. For example, if a, b and c are distinct complex numbers, the configuration $(a,b,b,c) \in Sym^4(\mathbb{C})$ corresponds to the partition $1+1+2$.

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For λ a partition of k, define $w_\lambda(X)$ to be the subspace of $Sym^k(X)$ of points whose corresponding partition is λ. Given a partition $\lambda = (a_1 + \cdots + a_n)$ of k, let $1^j\lambda$ be the partition $(1+1+\cdots+1+a_1+\cdots+a_n)$ of $j+k$ obtained by adding j ones to λ. Note that the space $w_{1^j}(X)$ is the configuration space of j unordered distinct particles in M. We can now state Conjectures G and H.

Theorem 1.1 *(Conjecture G). For j sufficiently large compared to i, we have that $H_i(w_{1^j}(\mathbb{C}P^2);\mathbb{Q}) = \mathbb{Q}$ for $i = 0, 2, 4, 7, 9, 11$ and 0 otherwise.*

Theorem 1.2 *(Conjecture H). The colimit $\operatorname{colim}_{j\to\infty} H_i(w_{1^j 2}(\mathbb{C}^d);\mathbb{Q})$ is eventually periodic in i.*

Vakil and Wood also predicted in Formula 1.50 of [10] that the groups appearing in Theorem 1.2 are as follows:

$$\operatorname*{colim}_{j\to\infty} H_i\big(w_{1^j 2}(\mathbb{C}^d);\mathbb{Q}\big) = \begin{cases} \mathbb{Q} & \text{if } i = 0 \\ \mathbb{Q}^2 & \text{if } i = 2(2k-1)d - 1 \text{ or } 4kd \text{ for } k \geqslant 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

This is not the case; they are instead equal to:

$$\operatorname*{colim}_{j\to\infty} H_i\big(w_{1^j 2}(\mathbb{C}^d);\mathbb{Q}\big) = \begin{cases} \mathbb{Q} & \text{if } i = 0 \\ \mathbb{Q}^2 & \text{if } i = k(2d-1) \text{ for } k \geqslant 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Thus the spaces $w_{1^j 2}(\mathbb{C}^d)$ are an example where Occam's razor for Hodge structures yields incorrect guesses for stable homology groups.

The organization of this note is as follows. In Section 2, we recall scanning and homological stability results of McDuff in [4] and Segal in [7]. Using these theorems as well as some rational homology calculations from [8] and [5], we prove Conjecture G in Section 3 and Conjecture H in Section 4.

We have been informed that Orsola Tommasi has an alternative proof of Conjecture H. She observed that the cohomology groups of $w_\lambda(\mathbb{C}^d)$ have pure weight. This allows one to determine the stable rational cohomology of $w_{1^j 2}(\mathbb{C}^d)$ from the corresponding element in the Grothendieck ring of varieties, which was computed in [10]. The case of $d = 1$ of Conjecture H has also been proven by Thomas Church, Jordan Ellenberg and Benson Farb (Proposition 4.4 of [2]) using the machinery of étale cohomology and L-functions. This is part of their general program relating homological stability results for configuration spaces to the combinatorics of polynomials over finite fields.

2. Scanning

Our primary tool for proving these conjectures is Theorem 1.1 of McDuff in [4] regarding the so-called scanning map. For M a manifold, let $\dot{T}M$ denote the fiberwise one-point compactification of the tangent bundle of M. Let $\Gamma_k^c(\dot{T}M)$ denote the space of compactly supported sections of $\dot{T}M \longrightarrow M$ of degree k (see [4] or [1] for the definition of degree) topologized with the compact-open topology. In [4] McDuff constructed a map $s : w_{1^j}(M) \longrightarrow \Gamma_k^c(\dot{T}M)$ and proved the following theorem.

Theorem 2.1. *The scanning map $s : w_{1^j}(M) \longrightarrow \Gamma_j^c(\dot{T}M)$ induces an isomorphism on homology groups H_i for $j \gg i$. If M is the interior of a compact connected manifold with non-empty boundary, then the scanning map induces an injection on all homology groups.*

The isomorphism portion of the above theorem is Theorem 1.1 of [4] and the injectivity statement is contained in the proof of Theorem 4.5 of [4]. One can find explicit ranges for this theorem. By Proposition A.1 of [7], for homology with integral coefficients $j \geqslant i/2$ suffices. By Theorem B of [6], for homology with rational coefficients $j \geqslant i$ suffices if the dimension of M is at least 3.

3. Conjecture G

In this section we prove Conjecture G. By Theorem 2.1 this amounts to computing the rational homology groups of a space of sections. In principle this can be done using the techniques introduced in [9]. However, we will first compare this space of sections with the space of continuous degree l maps $Map_l(\mathbb{C}P^2, S^4)$. This space is topologized with the compact open topology and the degree of a map is defined via the induced map on top dimensional homology. Because $\mathbb{C}P^2$ is compact the space of compactly supported sections is equal to the space of all sections. We use the following proposition, which follows from Proposition 3.5 of [1].

Proposition 3.1. *Let M be a compact orientable manifold of dimension n and let $\chi(M)$ denote its Euler characteristic. For all integers $k \neq \chi(M)/2$ and integers $l \neq 0$, $\Gamma_k^c(\dot{T}M)$ is rationally homotopy equivalent to $Map_l(M, S^n)$.*