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Isomorphism conjectures with proper coefficients ☆

Guillermo Cortiñas [a,*], Eugenia Ellis [b]

[a] Dep. Matemática-IMAS, FCEyN-UBA, Ciudad Universitaria Pab 1, 1428 Buenos Aires, Argentina
[b] Centro de Matemática, Facultad de Ciencias, Universidad de la República, Iguá 4225, 11400 Montevideo, Uruguay

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ABSTRACT

Let G be a group and $\mathcal{F}$ a nonempty family of subgroups of G, closed under conjugation and under subgroups. Also let E be a functor from small $\mathbb{Z}$-linear categories to spectra, and let A be a ring with a G-action. Under mild conditions on E and A one can define an equivariant homology theory $H^G(-, E(A))$ of G-simplicial sets such that $H^G_*(G/H, E(A)) = E(A \rtimes H)$. The strong isomorphism conjecture for the quadruple $(G, \mathcal{F}, E, A)$ asserts that if $X \to Y$ is an equivariant map such that $X^H \to Y^H$ is an equivalence for all $H \in \mathcal{F}$, then

$$H^G(X, E(A)) \to H^G(Y, E(A))$$

is an equivalence. In this paper we introduce an algebraic notion of $(G, \mathcal{F})$-properness for G-rings, modeled on the analogous notion for G-C^*-algebras, and show that the strong $(G, \mathcal{F}, E, P)$ isomorphism conjecture for $(G, \mathcal{F})$-proper P is true in several cases of interest in the algebraic K-theory context.

1. Introduction

Let G be a group; a *family* of subgroups of G is a nonempty family $\mathcal{F}$ closed under conjugation and under taking subgroups. If $\mathcal{F}$ is a family of subgroups of G, then a G-simplicial set X is called a *$(G, \mathcal{F})$-complex* if the stabilizer of every simplex of X is in $\mathcal{F}$. The category of G-simplicial sets can be equipped with a closed model structure where an equivariant map $X \to Y$ is a weak equivalence (resp. a fibration) if $X^H \to Y^H$ is a weak equivalence (resp. a fibration) for every $H \in \mathcal{F}$ (see Section 2); $(G, \mathcal{F})$-complexes are the cofibrant objects in this model structure (Remark 2.5). By a general construction of Davis and Lück (see [9]) any functor E from the category $\mathbb{Z}-\mathrm{Cat}$ of small $\mathbb{Z}$-linear categories to the category Spt of spectra which sends category equivalences to equivalences of spectra gives rise to an equivariant homology theory of G-spaces $X \mapsto H^G(X, E(R))$ for each unital ring R with a G-action (unital G-ring, for short), such that if $H \subset G$ is a subgroup, then

$$H^G_*(G/H, E(H)) = E_*(R \rtimes H) \tag{1.1}$$

is just E_* evaluated at the crossed product. The *strong isomorphism conjecture* for the quadruple $(G, \mathcal{F}, E, R)$ asserts that $H^G(-, E(R))$ sends $(G, \mathcal{F})$-equivalences to weak equivalences of spectra. The strong isomorphism conjecture is equivalent to the assertion that for every G-simplicial set X the map

$$H^G(cX, E(R)) \to H^G(X, E(R)) \tag{1.2}$$

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* Corresponding author.
E-mail addresses: gcorti@dm.uba.ar (G. Cortiñas), eugenia@cmat.edu.uy (E. Ellis).
URL: http://mate.dm.uba.ar/~gcorti (G. Cortiñas).

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induced by the $(G,\mathcal{F})$-cofibrant replacement $cX \to X$ is a weak equivalence. The weaker *isomorphism conjecture* is the particular case when X is a point; it asserts that if $\mathcal{E}(G,\mathcal{F}) \overset{\sim}{\twoheadrightarrow} pt$ is the cofibrant replacement then the map

$$H^G\big(\mathcal{E}(G,\mathcal{F}), E(R)\big) \to H^G\big(pt, E(R)\big) \tag{1.3}$$

called the *assembly map*, is an equivalence of spectra. This formulation of the conjecture is equivalent to that of Davis–Lück, [9] which is given in terms of topological spaces (see Proposition 2.3 and paragraph 2.6). One can more generally formulate the conjecture with coefficients in additive categories with a G-action [2], but we shall limit ourselves to rings here.

In this paper we are primarily concerned with the strong isomorphism conjecture for nonconnective algebraic K-theory – denoted K in this paper – homotopy algebraic K-theory KH, and Hochschild and cyclic homology HH and HC. Our main results are outlined in Theorem 1.4 below. First we need to explain the terms "excisive" and "proper" appearing in the theorem. Let $E:\mathrm{Rings} \to \mathrm{Spt}$ be a functor; we say that a not necessarily unital ring A is E-*excisive* if whenever $A \to R$ is an embedding of A as a two-sided ideal in a unital ring R, the sequence

$$E(A) \to E(R) \to E(R/A)$$

is a homotopy fibration. Unital rings are E-excisive for all functors E considered in Theorem 1.4; thus the theorem remains true if "unital" is substituted for "excisive". By a result of Weibel [34], homotopy algebraic K-theory satisfies excision; this means that every ring is KH-excisive. Wodzicki characterized excision for Hochschild and cyclic homology in terms of H-unitality [35]. By results of Suslin and Wodzicki, a ring A is excisive for rational K-theory if and only if $A \otimes \mathbb{Q}$ is H-unital (see [30] for the if part and [35] for the only if part); K-excisive rings were characterized by Suslin in [29]. Under mild assumptions on E (Standing Assumptions 3.3.2), which are satisfied by all the examples considered in Theorem 1.4, one can make sense of $H^G(-, E(A))$ for not necessarily unital, E-excisive A (see Section 3). The ring $\mathbb{Z}^{(X)}$ of polynomial functions on a locally finite simplicial set X which are supported on a finite simplicial subset, and the ring $C_{\mathrm{comp}}(|X|, \mathbb{F})$ of compactly supported continuous functions with values in $\mathbb{F} = \mathbb{R}, \mathbb{C}$ are unital if and only if X is finite, and are E-excisive for all X and all the functors E of Theorem 1.4; they are $(G,\mathcal{F})$-proper whenever X is a $(G,\mathcal{F})$-complex. In general if X is a locally finite simplicial set with a G-action and A is a G-ring, then A is called *proper* over X if it carries a $\mathbb{Z}^{(X)}$-algebra structure which is compatible with the action of G and satisfies $\mathbb{Z}^{(X)} \cdot A = A$. We say that A is $(G,\mathcal{F})$-*proper* if it is proper over a $(G,\mathcal{F})$-complex.

Theorem 1.4. *Let G be a group, $\mathcal{F}$ a family of subgroups, $E:\mathbb{Z}-\mathrm{Cat} \to \mathrm{Spt}$ a functor, and P an E-excisive, $(G,\mathcal{F})$-proper G-ring. The strong isomorphism conjecture for the quadruple $(G,\mathcal{F},E,P)$ is satisfied in each of the following cases.*

i) $E = KH$.
ii) $E = K$ *and P is proper over a 0-dimensional $(G,\mathcal{F})$-complex.*
iii) $E = K$, *$\mathcal{F}$ contains all the cyclic subgroups of G and P is a $\mathbb{Q}$-algebra.*
iv) $E = K \otimes \mathbb{Q}$ *and $\mathcal{F}$ contains all the cyclic subgroups of G.*

Theorem 13.1.1 proves that part i) of the theorem above holds for any functor $E:\mathbb{Z}-\mathrm{Cat} \to \mathrm{Spt}$ satisfying certain properties, including excision; the fact that KH satisfies them is the subject of Section 5. We prove in Theorem 11.6 that part ii) of the theorem holds for any E satisfying the standing assumptions; that they hold for K-theory is established in Proposition 4.3.1. Parts iii) and iv) are the content of Theorem 13.2.1; their proof uses the fact, established in Proposition 7.6, that cyclic homology satisfies the strong isomorphism conjecture with coefficients in arbitrary H-unital G-rings. The latter proposition generalizes a result of Lück and Reich [21], who proved it for unital rings with trivial G-action.

The concept of properness used in this article is a discrete, algebraic translation of the analogous concept of proper G-C^*-algebra. By a result of Guentner, Higson and Trout, the full C^*-crossed product version of the Baum–Connes conjecture with coefficients holds whenever the coefficient algebra is a proper G-C^*-algebra [12]. This result is a basic fact behind the Dirac-dual Dirac method that was used, for example, in the proof of the Baum–Connes conjecture for a-T-menable groups [13]. It is also at the basis of recent work of Meyer and Nest [22–24] in which the conjecture and the Dirac method are recast in terms of triangulated categories. Theorem 1.4 will be used in [5] to prove the Farrell–Jones conjecture for the K-theory of the group algebra $\mathcal{K}[G]$ with coefficients in the ring $\mathcal{K}$ of compact operators in an infinite dimensional, separable Hilbert space when the group G is a-T-menable in the sense of Gromov. We expect it can similarly be used to prove other instances of the isomorphism conjecture for (homotopy) algebraic K-theory. In the current paper, we use Theorem 1.4 to prove the following theorem, which identifies the assembly map (1.3) as the connecting map in a functorial excision sequence. The latter sequence is a pure algebraic analogue of a construction given in the book by Cuntz, Meyer and Rosenberg in the C^*-algebraic context [8, §5.3].

Theorem 1.5. *Let G be a group and $\mathcal{F}$ a family of subgroups. Then there is a functor which assigns to each G-ring A a G-ring $\mathfrak{F}^\infty A = \mathfrak{F}^\infty(\mathcal{F}, A)$ equipped with an exhaustive filtration by G-ideals $\{\mathfrak{F}^n A\colon n \geqslant 0\}$, and a natural transformation $A \to \mathfrak{F}^0 A$, which, if E is as in Theorem 1.4 and A is E-excisive, have the following properties.*