Journal of Pure and Applied Algebra 218 (2014) 1264–1290

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Generic residual intersections and intersection numbers of movable components

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ARTICLE INFO

Article history:
Received 23 October 2012
Received in revised form 5 August 2013
Available online 15 November 2013
Communicated by C.A. Weibel

MSC:
Primary: 14C17; secondary: 13A30; 13H15

ABSTRACT

Given a sequence $\mathbf{x}$ of elements of a commutative equidimensional noetherian ring R, cycles $z_i(\mathbf{x}, R)$ $(i \in \mathbb{N})$ in the cycle group of polynomial rings over R are defined by generic residual intersections. The study of these cycles gives new insight into the theory for excess intersections in projective space developed by Stückrad and Vogel, in particular concerning the contribution to the intersection cycle of embedded components not defined over the base field.

In this paper “ring” always means “commutative noetherian ring with unit element different from its zero element”. An R-module M over a ring R is called *equidimensional*, if $\dim R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m} = \dim_R M$ for all $\mathfrak{m} \in \max \operatorname{Supp}_R M$ and all $\mathfrak{p} \in \min \operatorname{Supp}_R M$ with $\mathfrak{p} \subseteq \mathfrak{m}$. M is called *catenary* if for all pairs (P, Q) of prime ideals $P, Q \in \operatorname{Supp}_R M$ with $P \subseteq Q$ all maximal prime ideal chains between P and Q have the same length.

An R-module M is said to be *strict equidimensional*, if M is equidimensional and catenary. M is strict equidimensional iff any two maximal chains in $\operatorname{Supp}_R M$ have the same length (namely $\dim_R M$). If R is catenary, M is strict equidimensional if and only if M is equidimensional. A finitely generated R-module M is (strict) equidimensional iff $R/\operatorname{Ann}_R M$ is (strict) equidimensional. If X is a Zariski closed subset of $\operatorname{Spec} R$, then M is called (*strict*) *equidimensional relative to* X, if $X \cap \operatorname{Supp}_R M \neq \emptyset$, $M_\mathfrak{m}$ is a (strict) equidimensional $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m} \in X \cap \operatorname{Supp}_R M$ and $\dim_{R_\mathfrak{m}} M_\mathfrak{m} = \dim_{R_\mathfrak{n}} M_\mathfrak{n}$ for all maximal ideals $\mathfrak{m}, \mathfrak{n} \in X \cap \operatorname{Supp}_R M$.

We note that $\operatorname{minAss}_R M = \min \operatorname{Supp}_R M$ for all R-modules M. If M is strict equidimensional, then $\dim_{R_P} M_P + \dim R/P = \dim_R M$ for all $P \in \operatorname{Supp}_R M$.

0. Introduction

The scope of this paper is to get a better understanding of the structure of the intersection cycles in the intersection theory developed by Stückrad and Vogel in [13] (see also [7]) for the intersection of equidimensional closed subschemes $X, Y \subset \mathbb{P}^n_K = \operatorname{Proj}(K[X_0, \dots, X_n])$ (K a field) without embedded components. This intersection algorithm produces over a certain purely transcendental field extension L of the base field K an intersection cycle supported in a finite collection $\mathcal{C}(X, Y)$ of subvarieties $C \subseteq X_L \cap Y_L$, where X_L, Y_L are the subvarieties of $\mathbb{P}^n_L$ obtained from X, Y by base extension. For $C \in \mathcal{C}(X, Y)$ the coefficient $j(X, Y; C)$ of C in this cycle is called *intersection number of X and Y in C*. Together with the

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0022-4049/$ – see front matter
http://dx.doi.org/10.1016/j.jpaa.2013.11.017

algorithm, these numbers are used to formulate and prove a theorem of Bézout also in the case where X and Y do not intersect properly (i.e. if the intersection $X \cap Y$ does not have the "expected" dimension):

$$\operatorname{degree}(X) \cdot \operatorname{degree}(Y) = \sum_{C \in \mathcal{C}(X,Y)} j(X, Y; C) \cdot \operatorname{degree}(C).$$

$\mathcal{C}(X, Y)$ might contain also the so-called "empty subvariety" $\emptyset$ which is defined by $X_0 = \cdots = X_n = 0$. By definition, $\operatorname{degree}(\emptyset) = 1$ and $\dim(\emptyset) = -1$. Moreover, in $\mathcal{C}(X, Y)$ non-K-rational elements can occur, i.e., subvarieties of $X_L \cap Y_L$ which are not defined over K (but over L, of course). We note that non-K-rational elements $C \in \mathcal{C}(X, Y)$ are called *irrational* or *movable* or *mobile* components.

Since this paper is based on Stückrad and Vogel's approach to an algebraic theory of (improper) intersections in $\mathbb{P}^n_K$ (K an arbitrary field), we briefly recall these constructions. For indeterminates u_{ij} $(0 \leqslant i, j \leqslant n)$ let L be the pure transcendental field extension $K(u_{ij})_{0\leqslant i,j\leqslant n}$. Then the Stückrad–Vogel cycle

$$z(X, Y) = z_0 + \cdots + z_{n+1} = \sum_{C \in \mathcal{C}(X,Y)} j(X, Y; C)[C]$$

on $X_L \cap Y_L$ is obtained by an intersection algorithm on the ruled join

$$J_L := J(X_L, Y_L) \subset \mathbb{P}^{2n+1}_L = \operatorname{Proj}\bigl(L[X_0, \ldots, X_n, Y_0, \ldots, Y_n]\bigr)$$

as follows:

Let Δ_L be the "diagonal" subspace of $\mathbb{P}^{2n+1}_L$ given by the equations

$$X_0 - Y_0 = \cdots = X_n - Y_n = 0$$

and let $H_i \subseteq J_L$ be the divisor given by the equation

$$\ell_i := \sum_{j=0}^{n} u_{ij}(X_j - Y_j) = 0.$$

Then one defines inductively cycles ρ_k and z_k by setting $\rho_0 := [J_L]$ and, if ρ_k is already defined, by decomposition of the intersection

$$\rho_k \cap H_k = z_{k+1} + \rho_{k+1} \quad (0 \leqslant k \leqslant \dim J_L),$$

where the support of z_{k+1} lies in Δ_L and no component of ρ_{k+1} is contained in Δ_L. It follows that z_k is a $(\dim J_L - k)$-cycle on $X_L \cap Y_L \cong J_L \cap \Delta_L$ or zero.

Van Gastel [10] showed that the intersection theories of Fulton and MacPherson and of Stückrad and Vogel lead to equivalent results in the case where they both apply and that a K-rational subvariety C of $X_L \cap Y_L$ occurs in $z(X, Y)$ if and only if C is a distinguished variety of the intersection of X and Y in the sense of Fulton [9]. Therefore, the K-rational components of $z(X, Y)$ together with their intersection numbers are geometrically well understood. Conversely, by the Stückrad–Vogel algorithm, the distinguished varieties in the intersection theory of Fulton–MacPherson can be computed together with their intersection numbers using standard computer algebra packages. (If $C \in \mathcal{C}(X, Y)$ is K-rational, then its intersection number $j(X, Y; C)$ can be described as a coefficient of a Hilbert polynomial, see, for example, the μ-multiplicity of [2], the c_0-multiplicity of [3] or the j-multiplicity of [7, Chapter 6.1].)

As a consequence of the main result of this paper we are able to give a precise description of the non-K-rational (or movable) part of the intersection cycle $z(X, Y)$, see Section 5. For this we have to work in a more general context: The homogeneous coordinate ring of the ruled join $J(X, Y)$ of X and Y is replaced by an arbitrary ring R, the sequence $X_0 - Y_0, \ldots, X_n - Y_n$ of the generators of the diagonal ideal by a sequence $\mathbf{x} := (x_1, \ldots, x_r)$ of elements of R and the field extension $L = K(u_{ij})_{0\leqslant i,j\leqslant n}$ by the ring extensions $S^{(i)} := R[u_{11}, \ldots, u_{1r}, \ldots, u_{i1}, \ldots, u_{ir}]$, $i = 1, 2, \ldots$. We emphasize that in contrast to [13] now the intersection "algorithm" never stops, but there is an $i_1 \in \mathbb{N}^+$ such that for all $i \geqslant i_1$ the non-zero cycles $z_i(\mathbf{x}, R)$ defined in this general context stabilize in the sense specified in Remark 4.1.

The aim of this paper is to study these abstract algebraic cycles $z_i(\mathbf{x}, R)$, which are of their own interest and give new insight into the original intersection cycle $z(X, Y)$. It turns out that to each movable component of intersection, say C', there can be assigned precisely one of the K-rational (or *fixed*) components, say C, in which it is embedded, i.e. $C' \subset C$, and one has $j(X, Y; C') \leqslant j(X, Y; C)$.

1. The cycles $z_i(\mathbf{x}, R)$

Let R be a ring. In order to explain the construction of the cycle $z_i(\mathbf{x}, R)$, we recall some notation (see, e.g., [12] for standard notations).

If U is a submodule of an R-module M and J an ideal in R, let

$$U :_M \langle J \rangle := \bigl\{ m \in M \bigm| J^t \cdot m \subseteq U \text{ for some } t \in \mathbb{N} \bigr\}.$$