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Two-dimensional regularity and exactness ☆

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ABSTRACT

We define notions of regularity and (Barr-)exactness for 2-categories. In fact, we define *three* notions of regularity and exactness, each based on one of the three canonical ways of factorising a functor in **Cat**: as (surjective on objects, injective on objects and fully faithful), as (bijective on objects, fully faithful), and as (bijective on objects and full, faithful). The correctness of our notions is justified using the theory of lex colimits [12] introduced by Lack and the second author. Along the way, we develop an abstract theory of regularity and exactness relative to a kernel–quotient factorisation, extending earlier work of Street and others [24,3].

1. Introduction

This paper is concerned with two-dimensional generalisations of the notions of *regular* and *Barr-exact* category. There are in fact a plurality of such generalisations and a corresponding plurality of articles exploring these generalisations—see [2,3,7,8,10,11,22,23], for example. Examining this body of work, one finds a clear consensus as to the form such generalised notions should take: one considers a 2-category or bicategory equipped with finite limits and with a certain class of colimits, and requires certain "exactness" conditions to hold between the finite limits and the specified colimits. For generalised regularity, the finite limits are used to form the "kernel" of an arrow; the specified colimits are just those needed to form "quotients" of such kernels; and the exactness conditions ensure that the process of factoring an arrow through the quotient of its kernel gives rise to a well-behaved factorisation system on the 2-category or bicategory in question. For generalised Barr-exactness, the finite limits are used to specify "congruences" (the maximal finite-limit structure of which all "kernels" are instances); the specified colimits are those required to form quotients of congruences; and the exactness conditions extend those for regularity by demanding that every congruence be "effective"—the kernel of its quotient.

Whilst this general schema for regularity and Barr-exactness notions is clear enough (and as we shall soon see, makes sense in a more general setting than just that of 2-categories), the details in individual cases are less so. The main complication lies in ascertaining the right exactness conditions to impose between the finite limits and the specified colimits. Previous authors have done so in an essentially ad hoc manner, guided by intuition and a careful balancing of the opposing constraints of sufficient examples and sufficient theorems. The first contribution of this paper is to show that any instance of the schema admits a canonical, well-justified choice of exactness conditions, which automatically implies many of the desirable properties that a generalised regular or Barr-exact category should have.

We obtain this canonical choice from the theory of *lex colimits* developed by the second author and Lack in [12]. This is a framework for dealing with $\mathcal{V}$-categorical structures involving limits, colimits, and exactness between the two; one of the key insights is that, for a given class of colimits, the appropriate exactness conditions to impose are just those which

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hold between finite limits and the given colimits in the base $\mathcal{V}$-category $\mathcal{V}$; more generally, in any "$\mathcal{V}$-topos" (lex-reflective subcategory of a presheaf $\mathcal{V}$-category). Applied in the case $\mathcal{V} = \mathbf{Set}$, this theory justifies the exactness conditions for the notions of regular and Barr-exact category as well as those of extensive, coherent or adhesive categories; applied in the case $\mathcal{V} = \mathbf{Cat}$, it will *provide* us with the exactness conditions for our generalised regularity and Barr-exactness notions.

The second contribution of this paper is to study in detail three particular notions of two-dimensional regularity and Barr-exactness. As we have said, there are a range of such notions; in fact, there is one for each well-behaved orthogonal factorisation system on **Cat**, and the three examples we consider arise from the following ways of factorising a functor:

(i) (surjective on objects, injective on objects and fully faithful);
(ii) (bijective on objects, fully faithful); and
(iii) (bijective on objects and full, faithful).

Of course, many other choices are possible—interesting ones for further investigation would be (final, discrete opfibration) [26] and (strong liberal, conservative) [8]—but amongst all possible choices, these three are the most evident and in some sense the most fundamental. For (i), the notion of regularity we obtain is more or less that defined in [23, §1.19]; the exactness conditions amount simply to the stability under pullback of the quotient morphisms. In the case (ii), we obtain the folklore construction of (bijective on objects, fully faithful) factorisations via the codescent object of a higher kernel; see [25, §3], for example. However, the exactness conditions required do *not* simply amount to stability under pullback of codescent morphisms; one must also impose the extra condition that, if $A \to B$ is a codescent morphism, then so also is the diagonal map $A \to A \times_B A$. This condition, forced by the general theory of [12], has not been noted previously and is moreover, substantive: for example, the category **Set**, seen as a locally discrete 2-category, satisfies all the other prerequisites for regularity in this sense, but *not* this final condition. Finally, the regularity notion associated with the factorisation system (iii) appears to be new, although an abelian version of it is considered in [14]. The corresponding analogues of Barr-exactness for (i), (ii) and (iii) supplement the regularity notions by requiring effective quotients of appropriate kinds of congruences: for (i), these are the congruences discussed in [23, §1.8]; for (ii) they are the *cateads* of [7]; whilst for (iii), they are internal analogues of the notion of category equipped with an equivalence relation on each hom-set, compatible with composition in each variable.

We find that there are many 2-categories which are regular or exact in the senses we define. **Cat** is so essentially by definition; and this implies the same result for any presheaf 2-category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]$. The category of algebras for any 2-monad on **Cat** which is *strongly finitary* in the sense of [16] is again regular and exact in all senses; which encompasses such examples as the 2-category of monoidal categories and strict monoidal functors; the 2-category of categories equipped with a monad; the 2-category of categories with finite products and strict product-preserving functors; and so on. Another source of examples comes from internal category theory. If $\mathcal{E}$ is a category with finite limits, then $\mathbf{Cat}(\mathcal{E})$ is *always* regular and exact relative to the factorisation system (ii); if $\mathcal{E}$ is moreover regular or Barr-exact in the usual 1-categorical sense, then $\mathbf{Cat}(\mathcal{E})$ will be regular or exact relative to (i) and (iii) also. Finally, we may combine the above examples in various ways: thus, for instance, the 2-category of internal monoidal categories in any Barr-exact category $\mathcal{E}$ is regular and exact in all three senses.

As we mentioned in passing above, there is nothing inherently two-dimensional about the schema for generalised regularity and exactness; it therefore seems appropriate to work—at least initially—in a more general setting. Over an arbitrary enrichment base $\mathcal{V}$, one may define a notion of *kernel–quotient system* whose basic datum is a small $\mathcal{V}$-category $\mathcal{F}$ describing the shape of an "exact fork": the motivating example takes $\mathcal{V} = \mathbf{Set}$ and $\mathcal{F} = \bullet \rightrightarrows \bullet \to \bullet$. Given only this $\mathcal{F}$, one may define analogues of all the basic constituents of the theory of regular and Barr-exact categories; the particular examples of interest to us will arise from three suitable choices of $\mathcal{F}$ in the case $\mathcal{V} = \mathbf{Cat}$. The theory of kernel–quotient systems was first investigated by Street in unpublished work [24], and developed further in a preprint of Betti and Schumacher [3]; a published account of some of their work may be found in [10]. As indicated above, the new element we bring is the use of the ideas of [12] to justify the exactness conditions appearing in the notions of $\mathcal{F}$-regularity and $\mathcal{F}$-exactness.

Finally, let us remark on what we do *not* do in this paper. All the two-dimensional exactness notions we consider will be strict 2-categorical ones; thus we work with 2-categories rather than bicategories, 2-functors rather than homomorphisms, weighted 2-limits rather than bilimits, and so on. In other words, we are working within the context of **Cat**-enriched category theory; this allows us to apply the theory of [12] directly, and noticeably simplifies various other aspects of our investigations. There are bicategorical analogues of our results, which are conceptually no more difficult but are more technically involved; we have therefore chosen to present the 2-categorical case here, reserving the bicategorical analogue for future work.

We now describe the contents of this paper. We begin in Section 2 by defining kernel–quotient systems and developing aspects of their theory; as explained above, this material draws on [24] and [3]. We do not yet define the notions of regularity and exactness relative to a kernel–quotient system $\mathcal{F}$; before doing so, we must recall, in Section 3, the relevant aspects of the lex colimits of [12]. This then allows us, in Section 4, to complete the definitions of $\mathcal{F}$-regularity and $\mathcal{F}$-exactness, and to show that many of the desirable properties of an $\mathcal{F}$-regular or $\mathcal{F}$-exact category follow already at this level of generality.

This completes the first main objective of the paper; in Section 5, we commence the second, by introducing the two-dimensional kernel–quotient systems corresponding to (i)–(iii) above, and studying their properties. In Section 6, we describe