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Gorenstein conditions over triangular matrix rings

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ABSTRACT

A ring is left Gorenstein regular if the classes of left modules with finite projective dimension and finite injective dimension coincide and the injective and projective finitistic left dimensions are finite. Let A and B be rings and U a (B, A)-bimodule such that ${}_BU$ has finite projective dimension and U_A has finite flat dimension. In this paper we characterize when the ring $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ is left Gorenstein regular and, over such rings, when a left T-module is Gorenstein projective or Gorenstein injective. As applications of these results, we characterize when T is left CM-free and give a necessary condition for existence of an infinite cardinal λ such that each Gorenstein projective module is a direct sum of $\lambda^{<}$-generated modules.

1. Introduction

A ring (not necessarily commutative with unit) R is said to be Iwanaga–Gorenstein if it is two sided noetherian and the injective dimensions of ${}_RR$ and R_R are finite. Extending this notion to a Grothendieck category, in [10], such a category $\mathcal{C}$ is called Gorenstein if the classes of objects with finite projective dimension and finite injective dimension coincide, the injective and projective finitistic dimensions are finite and $\mathcal{C}$ has a generator with finite projective dimension. When we apply this definition to the category R-Mod of left modules over the ring R we obtain a one-sided Gorenstein notion without the noetherian condition. We call these rings left Gorenstein regular since, as it is proved in [10, Theorem 2.28], the global left Gorenstein projective and injective dimensions are finite. Of course, each Iwanaga–Gorenstein ring is left and right Gorenstein regular and the converse is true precisely when the ring is two sided noetherian. Moreover, many results that are true in the category of modules over an Iwanaga–Gorenstein ring are extended in [10] to a left Gorenstein regular ring (or, more generally, to a Gorenstein category), specially those which refer to Gorenstein projective and Gorenstein injective modules. Recall that a left R-module M is said to be Gorenstein projective (resp. injective) if there exists an exact complex

$$\mathbf{C} : \cdots \longrightarrow C^{n-1} \xrightarrow{\partial_{\mathbf{C}}^{n-1}} C^n \xrightarrow{\partial_{\mathbf{C}}^{n}} C^{n+1} \longrightarrow \cdots$$

such that C^i is projective (resp. injective) for each integer i, $\mathrm{Hom}(\mathbf{C}, P)$ is exact for each projective module P (resp. $\mathrm{Hom}(E, \mathbf{C})$ is exact for each injective module E) and $M \cong \mathrm{Ker}\, \partial^0$ (see [11]).

The main objective of this paper is to study these Gorenstein notions (Gorenstein regular rings, Gorenstein projective and Gorenstein injective modules) over triangular matrix rings. In order to do this, we use the well known description of modules over triangular matrix rings. More precisely, if A and B are rings and U is a (B, A)-bimodule, any left T-module M, where $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$, is determined by the data $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, where M_1 is a left A-module, M_2 is a left B-module and

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$\varphi^M : U \otimes M_1 \to M_2$ is a B-morphism (see [13, Theorem 1.5]). Using this description we prove in Theorem 3.1 that, when ${}_BU$ has finite projective dimension and U_A has finite flat dimension, the ring T is left Gorenstein regular if and only if A and B are left Gorenstein regular. Then, we characterize Gorenstein projective left modules over left Gorenstein regular triangular matrix rings (see Theorem 3.5), extending [19, Theorem 1.1], where an analogous characterization is given for some triangular matrix Artin algebras. Dual arguments give the corresponding result for Gorenstein injective modules, see Theorem 3.8.

We finish the article with two applications of the results. First of all we study when the triangular matrix ring T is left CM-free, that is, when all Gorenstein projective modules are projective. This property has been treated, for instance, in [7] for Artin algebras and in [8] for commutative noetherian local rings. In Theorem 4.1 we prove that the triangular matrix ring T is left CM-free if and only if A and B are left CM-free. Secondly, we study when the class of all Gorenstein projective modules is a strong Kaplansky class, in the sense that there exists an infinite cardinal λ such that each Gorenstein projective module is a direct sum of $\lambda^<$-generated modules (a module is $\lambda^<$-generated if it can be generated by less than λ elements). CM-free rings and CM-finite Gorenstein Artin algebras (in the sense that there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective modules) are rings with this property (see [6]). In Corollary 4.6 we prove that if the class of all Gorenstein projective modules over a left Gorenstein regular ring R is a strong Kaplansky class, then so is the class of all Gorenstein projective modules over the triangular matrix ring $\begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$.

2. Preliminaries

In this paper, all rings will be with identity. Morphisms will operate on the right and the composition of $f : A \to B$ and $g : B \to C$ will be fg. Module will mean left module. Let R be any ring. For each left R-module M, $\operatorname{proj\,dim} M$, $\operatorname{inj\,dim} M$ and $\operatorname{flat\,dim} M$ will denote the projective, injective and flat dimensions of M respectively. Moreover, $\mathrm{FPD}(R)$ and $\mathrm{FID}(R)$ will be the finitistic projective and injective dimensions of R respectively. If M is any module, $\mathrm{Add}(M)$ will be the class of all modules isomorphic to direct summands of direct sums of copies of M.

Let R be a ring. A complex $\mathbf{C}$ of modules will be an unbounded complex, i.e.:

$$\mathbf{C} : \cdots \longrightarrow C^{n-1} \xrightarrow{\partial_{\mathbf{C}}^{n-1}} C^n \xrightarrow{\partial_{\mathbf{C}}^{n}} C^{n+1} \longrightarrow \cdots$$

If $\mathcal{C}$ is an abelian category and $\mathbf{f} : R\text{-Mod} \to \mathcal{C}$ is an additive covariant functor, $\mathbf{f}(\mathbf{C})$ will be the complex

$$\mathbf{f}(\mathbf{C}) : \cdots \longrightarrow \mathbf{f}(C^{n-1}) \xrightarrow{\mathbf{f}(\partial_{\mathbf{C}}^{n-1})} \mathbf{f}(C^n) \xrightarrow{\mathbf{f}(\partial_{\mathbf{C}}^{n})} \mathbf{f}(C^{n+1}) \longrightarrow \cdots$$

We shall say that $\mathbf{C}$ is $\mathbf{f}$-exact if $\mathbf{f}(\mathbf{C})$ is exact. When the complex $\mathbf{C}$ is understood, we shall omit the subscript of the differential map, and we shall simply write ∂^n for each $n \in \mathbb{Z}$.

Recall that a module M is said to be Gorenstein projective (resp. Gorenstein injective) if there exists an exact complex $\mathbf{P}$ consisting of projective (resp. injective) modules which is $\mathrm{Hom}_R(\_, C)$-exact (resp. $\mathrm{Hom}_R(C, \_)$-exact) for each projective (resp. injective) module C and such that $M = \operatorname{Ker} \partial_{\mathbf{P}}^0$.

Let $\mathcal{C}$ be a Grothendieck category. $\mathcal{C}$ is said to be Gorenstein if it satisfies:

(1) The classes of all objects with finite projective dimension and with finite injective dimension coincide.
(2) The finitistic projective and injective dimensions of $\mathcal{C}$ are finite.
(3) $\mathcal{C}$ has a generator with finite projective dimension.

These categories have been introduced in [10]. We are interested in those rings for which the corresponding category of modules is Gorenstein.

Definition 2.1. A ring R is said to be left Gorenstein regular if the category R-Mod is Gorenstein.

Each Iwanaga–Gorenstein ring (that is, a two sided noetherian ring with finite left and right self-injective dimensions) is left and right Gorenstein regular (see [11, Theorem 9.1.11]).

Left Gorenstein regular rings are called simply left Gorenstein in [5, Definition VII.2.5]. For Artin algebras, the name left Gorenstein is used when the algebra has finite left self-injective dimension, see [4], while Gorenstein Artin algebras are those with finite left and right self-injective dimensions (that is, the algebra is Iwanaga–Gorenstein). For Artin algebras, our notion of left Gorenstein regular coincides with that of Gorenstein algebra. That is, an Artin algebra is left Gorenstein regular if and only if it is Gorenstein. More generally, if R is a two sided noetherian ring, R is left Gorenstein regular if and only if R is Iwanaga–Gorenstein. This is an easy consequence of [11, Proposition 9.1.1]. We think that the name left Gorenstein regular is more appropriate since, as a consequence of [10, Theorem 2.28], a ring R is left Gorenstein regular if and only if the global Gorenstein projective and injective dimensions of the ring are finite.

The finiteness of the finitistic dimensions is redundant in the definition of left Gorenstein regular ring, as it was proved in [5, Corollary VII.2.6]: